

*Barry Simon*  
Caltech

## **The lost proof of Loewner's Theorem**

### ABSTRACT

A real-valued function,  $F$ , on an interval  $(a,b)$  is called matrix monotone if  $F(A) < F(B)$  whenever  $A$  and  $B$  are finite matrices of the same order with eigenvalues in  $(a,b)$  and  $A < B$ . In 1934, Loewner proved the remarkable theorem that  $F$  is matrix monotone if and only if  $F$  is real analytic with continuations to the upper and lower half planes so that  $\text{Im } F > 0$  in the upper half plane.

This deep theorem has evoked enormous interest over the years and a number of alternate proofs. There is a lovely 1954 proof that seems to have been "lost" in that the proof is not mentioned in various books and review article presentations of the subject, and I have found no references to the proof since 1960. The proof uses continued fractions.

I'll provide background on the subject and then discuss the lost proof and a variant of that proof which I've found, which even avoids the need for estimates, and proves a stronger theorem.