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Representations of the symmetric groups and related groups and their combinatorics

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Since early on in the representation theory of finite groups, the representations of the symmetric groups S_n have played an important rôle. Starting with the classification of the irreducible complex characters of S_n by Frobenius (1900), which is closely related with the theory of symmetric functions, there has been an intimate relation between algebraic and representation theoretic properties and combinatorial questions. Partitions of n (i.e., sum decompositions of n where the order of the summands doesn't matter) label the irreducible representations in a natural way, and a recurring theme is the determination of representation theoretical data by combinatorial algorithms on the partitions.

Of particular interest are the dimensions of the S_n-representations, their branching behaviour w.r.t. restriction to S_{n-1} and the result of tensoring with the sign-representation. To all these questions there are well-known answers available if the representations are defined over a field of characteristic 0; nice combinatorial descriptions are given via branching and conjugation of the partitions of n labelling these representations.

It turned out that for p-modular representations (i.e. those defined over a field of characteristic p) the corresponding problems are much harder. There is great interest in such questions as there are strong connections between the symmetric groups and their representations and related groups such as the alternating groups or the covering groups of these groups, and also strong relations e.g. to representations of the general linear groups and Hecke algebras and their quantum analogues.

In recent years, important progress has been achieved with Kleshchev's Branching Theorems and the solution of the long-standing Mullineux Conjecture. Thus the Mullineux map (a rather complicated p-analogue of conjugation for p-regular partitions) provides the combinatorial answer to the question on the tensor product with the sign-representation for p-modular irreducible S_n-representations. The Mullineux map has motivated the definition of Mullineux symbols and more recently residue symbols, which may be viewed as a p-analogue of the well-known Frobenius symbols for partitions. For his modular branching results, Kleshchev has introduced the important new combinatorial concepts of good and normal nodes of a partition; the corresponding p-good Young graph coincides with the crystal graph occurring in the work of Kashiwara. As a first application of the residue symbols it was shown (in joint work with J. Olsson) that these behave well w.r.t. p-branching and p-conjugation simultaneously, thus allowing to provide a short proof of the combinatorial conjecture to which the Mullineux Conjecture had been reduced by Kleshchev and then proved by him and Ford in a long paper.

The better understanding of the Mullineux map has also opened up the road to investigating modular irreducible representations of the alternating groups A_n. Using the residue symbols and their properties, the branching behaviour of the A_n-representations has been studied in recent joint work with J. Olsson. The residue symbols have also been applied in the investigation of the Jantzen-Seitz partitions which label irreducible S_n-representations with irreducible restrictions to S_{n-1} (thus these partitions are the p-analogues of rectangle-shaped partitions); in particular, their p-cores have been determined. The Jantzen-Seitz-partitions have recently also come up in work by Foda et al. originating in problems of statistical mechanics, as well as in work by Brundan and Kleshchev on restrictions of representations from $GL(n)$ to $GL(n-1)$.