Mathematics 53.2, Fall 2008
Final Examination, 20 December
114points total

NAME: ______________________ **ID:** ______________________

GSI: ______________________

INSTRUCTIONS:
The examination has **TWO PARTS**, of 90 minutes each.

Part I (54 pts) is **MULTIPLE CHOICE** and no justification is necessary.
Record your answers by circling the appropriate letters on the **ANSWER SHEET** (last page).
Detach it from the exam paper, **WRITE DOWN YOUR NAME, ID AND GSI** on it
and pass it on towards the aisle when so instructed.
The answer sheets will be collected 90 minutes into the exam.

In **Part II** (60 pts), you must **justify your answers.**
All the work for a question must be on the respective sheet.
You may start work on Part II before the 90 minutes for Part I elapse
(but it is unwise to do so before you finish Part I).
You need not turn in the last sheet for rough work.

Part I: 18 questions in three groups. 3 points for correct answers, 1 point penalty for wrong answers. However, you will not receive a negative total on any group.

1. A (continuously differentiable) parametric curve C is given by $t \mapsto (r(t), \theta(t))$, in polar coordinates, with $0 \le t \le 1$. A formula for the arc length of C is

   (a) $\int_0^1 \sqrt{r'(t)^2 + \theta'(t)^2} dt$ (b) $\int_0^1 \sqrt{r'(t)^2 + r(t)^2 \cdot \theta'(t)^2} dt$

   (c) $\int_0^1 \sqrt{r(t)^2 + r'(t)^2 \cdot \theta(t)^2} dt$ (d) $\frac{1}{2}\int_0^1 r^2(t) dt$

2. If the parametric curve $t \mapsto \mathbf{r}(t)$, $a \le t \le b$, satisfies $\mathbf{r}(t) \cdot \mathbf{r}'(t) > 0$, it follows that

   (a) $|\mathbf{r}(t)|$ is constant (b) $|\mathbf{r}'(t)|$ is increasing

   (c) $|\mathbf{r}(t)|$ is increasing (d) None of the above.

3. The parametric curve $(x, y) = (f(t), g(t))$ must have a horizontal tangent at $(f(t_0), g(t_0))$ if

   (a) $f'(t_0) = 0$ (b) $g'(t_0) = 0$

   (c) $f'(t_0) = 0$ and $g'(t_0) \neq 0$ (d) $f'(t_0) \neq 0$ and $g'(t_0) = 0$

4. If the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^3$ satisfy $\mathbf{u} \times \mathbf{v} \neq 0$ and $\mathbf{u} \times \mathbf{w} \neq 0$, but $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w}) = 0$, then it follows that

   (a) The plane spanned by $\{\mathbf{u}, \mathbf{v}\}$ is orthogonal to that spanned by $\{\mathbf{u}, \mathbf{w}\}$.

   (b) $\mathbf{v} \perp \mathbf{w}$.

   (c) $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$.

   (d) $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ lie in the same plane.

5. If $f(x, y) = x^2 + y^3 + z^4$, then the tangent plane to the level surface $f(x, y, z) = 3$ at the point $(1, 1, 1)$ is given by the equation

   (a) $2x + 3y + 4z = 0$ (b) $2x + 3y + 4z = 9$

   (c) $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} = 9$ (d) $2x + 3y^2 + 4z^3 = 0$.

6. The function defined on $\mathbf{R}^2$ by $f(x, y) = \frac{x^2+xy+y^2}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 1$

   (a) Is differentiable everywhere

   (b) Is continuous everywhere, but not differentiable at $(0, 0)$

   (c) Has well-defined partial derivatives everywhere

   (d) Has well-defined and continuous partial derivatives everywhere

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7. The function $f(x, y) = x^2 + 3xy + y^2 + y^4$

   (a) Has a global minimum at $(0, 0)$

   (b) Has a local minimum at $(0, 0)$, but not a global minimum

   (c) Has a local maximum at $(0, 0)$

   (d) Has a saddle point at $(0, 0)$

8. If F and G are differentiable functions of (x, y, z), $G(P) = 0$ at some point P and ∇G does not vanish at P, then, subject to the constraint $G = 0$:

   (a) if F has a local extremum at P, then $\nabla F = \lambda \nabla G$ at P, for some λ.

   (b) if F has a local extremum at P, then $\nabla F \perp \nabla G$ at P.

   (c) we can be sure that F has a local extremum at P, if $\nabla F = \lambda \nabla G$ for some λ.

   (d) we can be sure that F has a local extremum at P, if $\nabla F \perp \nabla G$ at P.

9. If f is a differentiable function of x, y, which in turn are differentiable functions of u, v, then:

   (a) $f_u = f_x \cdot x_u + f_y \cdot y_u + f_v \cdot v_u$ (b) $f_u = f_x \cdot x_u + f_y \cdot y_u$

   (c) $f_u = f_x \cdot u_x + f_y \cdot u_y$ (d) $f_u = f_x \cdot x_u + f_y \cdot y_v$

10. The angle between the tangent plane to the surface $x^2 + y^2 + 2z^2 = 4$ at the point $(1, 1, 1)$ and the xy-plane is

    (a) $\pi/3$ (b) $\arcsin(\sqrt{2/3})$ (c) $\arccos(\sqrt{2/3})$ (d) $\arctan(\sqrt{2/3})$

11. Let f be a continuous 2-variable function on the disk D of radius 1 centered at $(0, 1)$. Which of the following expresses $\iint_D f(x, y)\, dA$?

    (a) $\int_{-\pi/2}^{\pi/2} \int_0^{2\sin\theta} f(r, \theta)\, r dr d\theta$ (b) $\int_0^{\pi} \int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\, r dr d\theta$

    (c) $\int_{-\pi/2}^{\pi/2} \int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\, r dr d\theta$ (d) $\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} f(x, y)\, dx dy$

12. Let R be the rectangle in $\mathbf{R}^2$ defined by $0 \le x \le \sqrt{3}$, $0 \le y \le 1$. In polar coordinates, the integral $\iint_R f\, dA$ of the continuous function f may be correctly written as:

    (a) $\int_0^{\pi/2} \int_0^{\sqrt{3}/\cos\theta} f \cdot r\, dr d\theta$

    (b) $\int_0^{\pi/3} \int_0^{\sqrt{3}/\cos\theta} f \cdot r\, dr d\theta + \int_{\pi/3}^{\pi/2} \int_0^{1/\sin\theta} f \cdot r\, dr d\theta$

    (c) $\int_0^{\pi/4} \int_0^{1/\cos\theta} f \cdot r\, dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{1/\sin\theta} f \cdot r\, dr d\theta$

    (d) $\int_0^{\pi/6} \int_0^{\sqrt{3}/\cos\theta} f \cdot r\, dr d\theta + \int_{\pi/6}^{\pi/2} \int_0^{1/\sin\theta} f \cdot r\, dr d\theta$

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13. Which of the following apply to vector field $\mathbf{F} = (x^2 - y^2)\mathbf{i} - 2xy\mathbf{j} + z^2\mathbf{k}$?

    (a) Its divergence vanishes

    (b) It is conservative and its curl vanishes

    (c) It is conservative, but its curl does not vanish

    (d) Its curl vanishes, but it is not conservative

14. Green's theorem for a plane region D enclosed by a simple, closed, positively oriented differentiable curve C, and a differentiable vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ on D, asserts that

    (a) $\iint_D (P_x + Q_y) dx dy = \oint_C \mathbf{F} \cdot d\mathbf{r}$ (b) $\iint_D (P_y + Q_x) dx dy = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds$

    (c) $\iint_D (Q_x - P_y) dx dy = \oint_C \mathbf{F} \cdot d\mathbf{r}$ (d) $\iint_D (Q_x - P_y) dx dy = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds$

    In (b) and (d), $\mathbf{n}$ is the outside normal vector and s the arc length parameter on C.

15. Let F be a continuous 2-component map, $(x, y) \mapsto (u, v) = F(x, y)$ taking the region D in $\mathbf{R}^2$ to some other region $F(D)$ in $\mathbf{R}^2$. The following formula for change of coordinates from (u, v) to (x, y) in the double integral of the continuous function f on $F(D)$,

$$\iint_{F(D)} f(u, v)\, dudv = \iint_D f\left(u(x, y), v(x, y)\right) \cdot (u_x v_y - u_y v_x)\, dxdy$$

(a) Applies to any F, f as described

(b) Applies whenever F is bijective and continuously differentiable

(c) As in (b), provided that, in addition, f is positive

(d) As in (b), provided that, in addition, $u_x v_y - u_y v_x$ is positive

16. Let $\mathbf{F}$ be a continuously differentiable vector field defined near the smooth surface S in $\mathbf{R}^3$, which is parametrised by $(u, v) \mapsto \mathbf{r}(u, v)$, $(u, v) \in D$, and is bounded by the piecewise smooth boundary curve C, oriented by the right-hand rule. Stokes' theorem asserts that:

(a) $\iint_D (\operatorname{curl} \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dudv = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

(b) $\iint_D (\operatorname{curl} \mathbf{F}) ||\mathbf{r}_u \times \mathbf{r}_v||\, dudv = \oint_C \mathbf{F} \cdot d\mathbf{r}$.

(c) $\iint_D \operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv = 0$.

(d) $\iint_D (\operatorname{curl} \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv = \oint_C \mathbf{F} \cdot \mathbf{n} ds$, where $\mathbf{n}$ is the normal vector to C and s the arc-length parameter on C.

17. Let S_+, S_- be the upper and lower unit hemispheres in $\mathbf{R}^3$. The fluxes of the vector field $\mathbf{F} = y^2 z^2 \mathbf{i} + x^2 z^2 \mathbf{j} + x^2 y^2 \mathbf{k}$ across S_+ and S_- are equal:

(a) if S_+ and S_- are both oriented using the normal pointing outside the unit sphere;

(b) if S_+ and S_- are both oriented using the upward normal;

(c) For the orientation determined by choosing θ first and ϕ second in spherical coordinates;

(d) for no choice of orientations.

18. For a continuously differentiable vector field $\mathbf{F}$ in an open region $D \subset \mathbf{R}^2$:

(a) $\operatorname{curl} \mathbf{F} = 0 \Rightarrow \mathbf{F}$ is conservative

(b) As in (a), but only under the additional assumption that $\operatorname{div} \mathbf{F} = 0$ as well

(c) As in (a), under the additional assumption that $\mathbf{D}$ is simply connected

(d) No statement above is correct.

Part II

Question 2 (15 pts)

Using Lagrange multipliers, find the maximum and minimum values for the function $f(z, y, z) = x^2 + y^2 + z^2$, when subject to the constraint $x^4 + y^4 + z^4 = 3$.

Caution: Mind the possible divisions by zero.

Question 3 (15 pts)
Draw a credible sketch of the polar curve C defined by $r(\theta) = \theta/2\pi$, for the range $0 \leq \theta \leq 6\pi$.
Write a parametric equation for the tangent line to C at the point $(1, 0)$.
Find the area of the region in the first quadrant bounded by the x-axis, the y-axis and the two arcs of C swept out as $2\pi \leq \theta \leq 2\pi + \pi/2$ and $4\pi \leq \theta \leq 4\pi + \pi/2$.

Question 4 (15 pts)
The *helicoid* is the surface in $\mathbf{R}^3$ parametrised by $(u, v) \mapsto u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$, $u \geq 0, v \in \mathbf{R}$. (Picture a spiral ramp winding upwards around the z-axis.) Consider the portion S of the helicoid given by $0 \leq u \leq 1$, $\alpha \leq v \leq \beta$.

(a) Express the surface area of S as a one-variable integral. (You need not solve the integral.)

(b) With $\alpha = 0, \beta = \pi$, compute $\iint_S \sqrt{x^2 + y^2}\, dA$.

(c) **Bonus, 5 points:** Evaluate the integral in Part (a). [Only if you have solved (a) and (b)]

Question 5 (15 pts)

(a) State the divergence theorem in $\mathbf{R}^3$, spelling out the assumptions and explaining the meaning of the terms in the formula.

(b) Compute the flux of the vector field

$$\mathbf{F} = (x + z^2 \arctan^2(y))\mathbf{i} - (y + \log(x^2 + 1) \sin z)\mathbf{j} + (z + 1)\mathbf{k}$$

across the hemisphere S given by $z = \sqrt{1 - x^2 - y^2}$, oriented upwards.

Hint: Compute $\operatorname{div} \mathbf{F}$ and find a good way to use the divergence theorem.

THIS PAGE IS FOR ROUGH WORK (not graded)

ANSWER SHEET FOR PART I

NAME: ____________________ **ID:** ____________________

GSI: ____________________

1.	a	b	c	d
2.	a	b	c	d
3.	a	b	c	d
4.	a	b	c	d
5.	a	b	c	d
6.	a	b	c	d
7.	a	b	c	d
8.	a	b	c	d
9.	a	b	c	d
10.	a	b	c	d
11.	a	b	c	d
12.	a	b	c	d
13.	a	b	c	d
14.	a	b	c	d
15.	a	b	c	d
16.	a	b	c	d
17.	a	b	c	d
18.	a	b	c	d