Math H53, Honors Multivariable Calculus (Kedlaya, fall 2002)
Second midterm exam, Thursday, November 7, 2002

The only permitted aid is one $8\frac{1}{2} \times 11$ sheet of paper (front and back). No other notes, calculator, or other assistance are permitted.

There are six problems, each on a separate page, plus an extra page if you need room for scratch work. However, please show all your work on the problem pages; you may continue on the back if you need more space. Work on the scratch page will not be graded.

The maximum number of points is 150.

Problem 1. This problem concerns a particle whose position vector at time t is given by $\mathbf{r}(t) = \langle t^2, 1 + t^3, -t \rangle$.

(a) Compute the velocity and acceleration of the particle at time t. (5 points)

(b) Compute the unit tangent and unit normal vectors at time t. (10 points)

(c) Compute the radius of the osculating circle to the path traced out by the particle at time t. (10 points)

Problem 2. Draw a contour plot that illustrates each of the following phenomena. Include at least five level curves, labeled with the values of the function there.

(a) A function $f(x, y)$ with two local maxima and a saddle point. (Mark the three critical points on the plot.) (5 points)

(b) A function $f(x, y)$ and a point at which f is maximized subject to $g(x, y) = 1$. (Mark the graph of $g(x, y) = 1$ and the point.) (5 points)

(c) A function $f(x, y)$ and a point at which $\nabla(f) = \langle 2, -1 \rangle$. (Mark the point on the plot and specify the scale, i.e., how long 1 unit is.) (10 points)

Problem 3. Define the function $f(x, y, z) = (x - y)\cos(xy + yz)$.

(a) Find the line perpendicular to the level surface $f(x, y, z) = 0$ at $(1, 1, -1)$. (10 points)

(b) Compute the first-order approximation to $f(1.01, 1.02, -1.01)$ around $(1, 1, -1)$. (5 points)

(c) Compute the second-order approximation to $f(1.01, 1.02, -1.01)$ around $(1, 1, -1)$. (10 points)

Problem 4. Consider z as a function of x and y given implicitly by the relation

$$xz^3 - xy^2 - yz + z^2 = 1.$$

(a) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x = y = z = 1$. (5 points)

(b) Suppose that x and y are themselves functions of u and v given by $x = u/v$ and $y = uv$. Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $u = v = z = 1$. (10 points)

(c) Set up a system of equations whose solutions include the triple (x, y, z) satisfying the given constraint, and also the additional constraint $x^2 + y^2 + z^2 = 3$, maximizing $f(x, y, z) = xyz$. Do not attempt to solve the system. (10 points)

Problem 5. Let $f(x,y) = y^3 - xy + x^2 - 2x + 2$, defined on the region R defined by the conditions $-1 \leq x \leq 1$ and $0 \leq y \leq 1$.

(a) Find all critical points of f within R. (10 points)

(b) Classify the critical points you found in (a) using the Second Derivative Test. (5 points)

(c) Find the absolute maximum and minimum of f on the region R. (10 points)

Problem 6. Consider a sphere of radius 1 centered at the origin. Remove from the sphere its intersection with an infinite cylinder of radius a whose axis is the z-axis. Let V be the remaining portion of the sphere.

(a) Set up a double integral in rectangular coordinates that computes the volume of V. (10 points)

(b) Set up a double integral in polar coordinates that computes the volume of V. (10 points)

(c) Compute the volume of V, as a function of a. (10 points)