THE "LANGLANDS PROGRAM" (ca. 1970) IS A UNIFIED FRAMEWORK FOR A HUGE RANGE OF RESULTS IN NUMBER THEORY — FROM QUADRATIC RECIPROCITY (ca. 1800) TO THE PROOF OF FERMAT'S LAST THEOREM (ca. 1995)

IT IS HARD, BECAUSE NUMBER THEORY IS HARD — AS CALCULUS IS POWERFUL! — AND EXPLAINING IT IS WAY BEYOND MY BAILIWICK AS A PHYSICIST.

FUNCTIONS ON X

QUANTIZING PATHS ON X

$$\Phi : \mathbb{R} \to X$$

$$I = \frac{1}{2}\int \left|\frac{d\Phi}{dt}\right|^2 dt$$

QUANTIZE

HAMILTONIAN

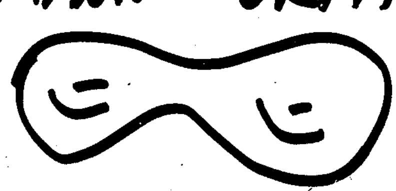
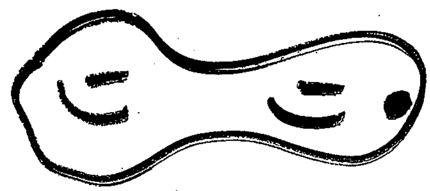
$H =$ LAPLACIAN

$$\Phi : \mathbb{R}^{1|2} \to X$$

DIFFERENTIAL FORMS

exterior derivative d adjt d^+

$$d^2 = (d^+)^2 = 0 \qquad \{d, d^+\} = \text{LAPLACIAN}$$

NUMBER THEORY	GEOMETRY
NUMBER FIELD SUCH AS $\mathbb{Q}$ = RATIONAL NUMBERS	RIEMANN SURFACE  C
	FIELD OF MEROMORPHIC FUNCTIONS ON C
PRIME NUMBER p	POINT $x \in C$  x
RATIONAL NUMBER q	MEROMORPHIC FUNCTION f
PRIME FACTORIZATION OF q	DESCRIBE f BY ZEROES AND POLES
$q = \prod_i p_i^{a_i}$	$f(z) = \prod_i (z - x_i)^{a_i}$
$a_i \in \mathbb{Z}$	$a_i \in \mathbb{Z}$
CLASS GROUP	JACOBIAN

WHEN TRANSLATED INTO GEOMETRY THE LANGLANDS PROGRAM INVOLVES INGREDIENTS THAT ARE PERHAPS A LITTLE MORE FAMILIAR, AT LEAST TO PHYSICISTS IN MY LINE OF WORK

GEOMETRIC LANGLANDS CORRESPONDENCE

C

$\rho : \pi_1(C) \to {}^L G_{\mathbb{C}}$

flat connection, minimum energy in gauge theory

Aharonov – Bohm effect

"$\mathcal{D}$-module" on the moduli space of G-bundles

related to two dimensional "current algebra," WZW model, etc.

with symmetry group G

FAMILIAR INGREDIENTS BUT COMBINED IN AN UNFAMILIAR WAY . . .

A LITTLE AS IF THE FAMILIAR PIECES OF A GAME OF CHESS WERE SCRAMBLED AND PLACED ON THE BOARD IN A RANDOM FASHION.

THIS HAS BEEN A PUZZLE FOR MANY YEARS, BUT RECENTLY THE PICTURE HAS BECOME CLEARER

A. KAPUSTIN and E. W.

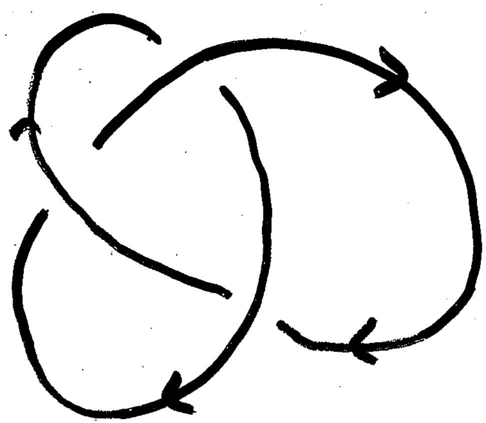
S. GUKOV and E. W.

NOW IF ONE AIMS TO RE-EXPRESS THE GEOMETRIC LANGLANDS PROGRAM IN A WAY THAT PHYSICISTS WILL FIND MORE NATURAL, AN OBVIOUS THOUGHT IS TO USE "GAUGE THEORY" — THE LANGUAGE OF ELEMENTARY PARTICLE THEORY — AND START ABOVE TWO DIMENSIONS ... MAYBE EVEN IN FOUR DIMENSIONS

THE THINKING HERE IS THAT IF TWO-DIMENSIONAL MATHEMATICAL PHYSICS WERE AN ADEQUATE FRAMEWORK FOR EXPRESSING GEOMETRIC LANGLANDS IN PHYSICAL TERMS, THIS WOULD HAVE BEEN OBVIOUS ALREADY.

ROUGH ANALOGY FROM 20 YEARS AGO

JONES POLYNOMIAL OF KNOTS



DESCRIBED ORIGINALLY VIA TWO-DIMENSIONAL INTEGRABLE SYSTEMS AND "CURRENT ALGEBRA"

ALTERNATIVE DESCRIPTION VIA THREE-DIMENSIONAL GAUGE THEORY

THERE IS ANOTHER FAIRLY OBVIOUS CLUE. THE LANGLANDS CORRESPONDENCE AND ITS GEOMETRIC ANALOG ARE STATED IN TERMS OF A PAIRING BETWEEN A SIMPLE (OR SEMISIMPLE) LIE GROUP G AND A DUAL GROUP ${}^L G$.

EXAMPLES

$$SU(2) \longleftrightarrow SO(3)$$

$$SU(N) \longleftrightarrow PSU(N) = SU(N)/\mathbb{Z}_N$$

$$SO(2N+1) \longleftrightarrow Sp(N)$$

$$E_8 \longleftrightarrow E_8$$

ONE WAY TO DESCRIBE THIS PAIRING INVOLVES ROOTS AND COROOTS ... MORE PRECISELY CHARACTERS AND COCHARACTERS

IF T AND ${}^LT$ ARE MAXIMAL TORI OF G AND ${}^LG$ THEN

$$\underbrace{\mathrm{HOM}(T, U(1))}_{\substack{\text{CHARACTER}\\ \text{LATTICE}\\ \text{OF } G}} = \underbrace{\mathrm{HOM}(U(1), {}^LT)}_{\substack{\text{COCHARACTER}\\ \text{LATTICE}\\ \text{OF } {}^LG}}$$

THERE IS ANOTHER STANDARD WAY TO EXPRESS IT:

AN IRREDUCIBLE REPRESENTATION R OF G IS IN NATURAL ONE-TO-ONE CORRESPONDENCE WITH A HOLOMORPHIC ${}^LG$-BUNDLE $E \to \mathbb{CP}^1$

(UP TO ISOMORPHISM ON EACH SIDE)

THE BASIS FOR THIS INTERPRETATION IS THAT THE REPRESENTATION R IS DETERMINED BY ITS "HIGHEST WEIGHT"

$$\rho : T \to U(1)$$

AND THE ${}^L G$-BUNDLE $E \to \mathbb{CP}^1$ IS "PULLED BACK" FROM THE USUAL LINE BUNDLE $\mathcal{L} = \mathcal{O}(1)$ OF STRUCTURE GROUP $U(1)$ VIA

$${}^L\rho : U(1) \to {}^L T \subset {}^L G$$

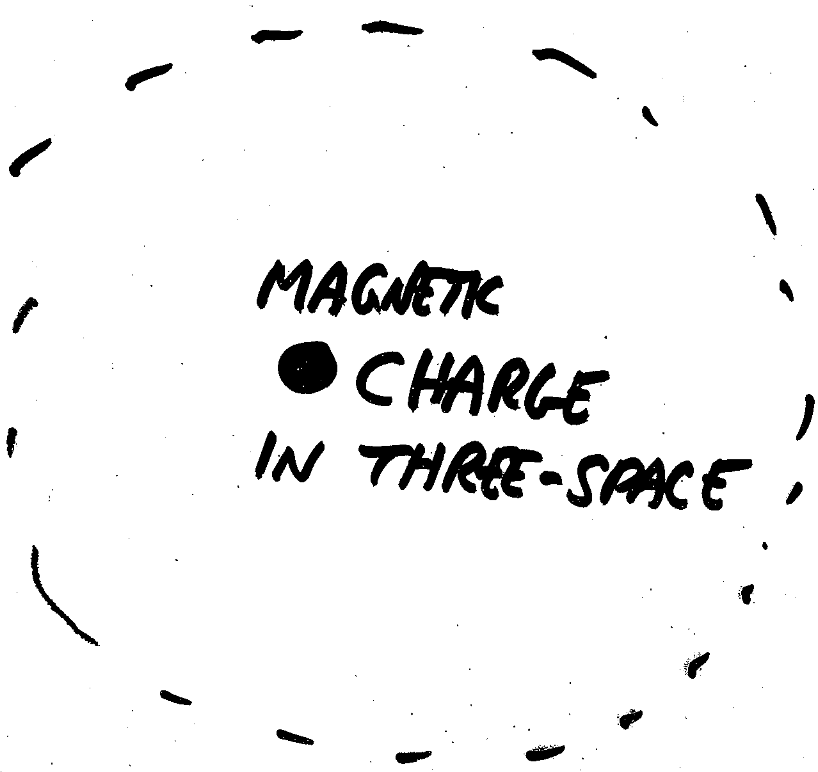
NOW IN THIS SECOND LANGUAGE THE CORRESPONDENCE BETWEEN G AND ^LG WAS DISCOVERED INDEPENDENTLY IN 1976 BY GODDARD, NUYTS, AND OLIVE.

THEIR STATEMENT IS THAT <u>ELEC</u>TRIC <u>CHAR</u>GE OF G IS CLASSIFIED LIKE MAG<u>NET</u>IC <u>CHAR</u>GE OF ^LG, AND VICE-VERSA.

STANDARD APPROACH IN GAUGE THEORY:

ELECTRIC CHARGE, ... MEASURED BY A REPRESENTATION OF G

MAGNETIC CHARGE ... MEASURED BY FIELDS AT INFINITY



"INFINITY" IS $S^2 \simeq \mathbb{C}P^2$

ONE STARTS WITH THE YANG-MILLS EQUATIONS ON S^2... BUT THE SOLUTIONS ARE HOLOMORPHIC BUNDLES OVER $\mathbb{C}P^1$.

S^2 OR $\mathbb{C}P^1$ BECAUSE PHYSICAL SPACETIME IS 3+1 - DIMENSIONAL

SOON AFTER, MONTONEN AND OLIVE TOOK THE NEXT STEP OF SAYING THAT, IN THE RIGHT CONTEXT, A GAUGE THEORY WITH GAUGE GROUP G WOULD BE EQUIVALENT QUANTUM MECHANICALLY TO A GAUGE THEORY WITH GAUGE GROUP ^LG.

WITH MANY SUBSEQUENT CLARIFICATIONS AND EXTENSIONS, THIS ULTIMATELY DEVELOPED (BY mid-1990's) INTO A CENTRAL THEME IN OUR UNDERSTANDING OF QUANTUM GAUGE THEORY AND STRING THEORY

... UNDERSTAND CONFINEMENT OF QUARKS

COUNT QUANTUM STATES OF A BLACK HOLE

UNIFY THE STRING THEORIES AND MAYBE THE FORCES OF NATURE

IT TOOK A WHILE BECAUSE AN ASSERTION ABOUT GAUGE THEORY THAT IS ONLY VALID QUANTUM MECHANICALLY IS DIFFICULT TO UNDERSTAND

ROUGH ANALOGY:

EQUIVALENCE BETWEEN DONALDSON THEORY OF FOUR-MANIFOLDS AND SEIBERG-WITTEN THEORY COMES FROM A SIMILAR QUANTUM DUALITY

THE CASE IN WHICH ELECTRIC-MAGNETIC DUALITY <u>DOES</u> MAKE SENSE CLASSICALLY IS THE FAMILIAR CASE OF ELECTROMAGNETISM, i.e. ABELIAN GAUGE THEORY WITH GAUGE GROUP U(1).

HERE, THIS "DUALITY" IS JUST THE SYMMETRY OF MAXWELL'S EQUATIONS UNDER EXCHANGE OF

$\mathbf{E}$ = ELECTRIC FIELD

$\mathbf{B}$ = MAGNETIC FIELD

i.e. IN VACUUM

$$\frac{\partial E}{\partial t} = \nabla \times B \qquad \nabla \cdot E = 0$$

$$\frac{\partial B}{\partial t} = -\nabla \times E \qquad \nabla \cdot B = 0$$

19^{th} CENTURY SYMMETRY

$$\vec{E} \to \vec{B}$$
$$\vec{B} \to -\vec{E}$$

RELATIVISTICALLY, WE COMBINE $\vec{E}$ & $\vec{B}$ TO A TWO-FORM

$$F = dt \wedge d\vec{x} \cdot \vec{E} + \frac{1}{2}\, d\vec{x} \cdot d\vec{x} \times \vec{B}$$

AND THEN MAXWELL'S EQUATIONS

IN VACUUM READ

$$0 = dF = d * F$$

$* =$ HODGE $*$

EVIDENT SYMMETRY

$$F \to *F$$

$$*F \to -F$$

TO PROCEED QUANTUM MECHANICALLY, WE NEED TO DERIVE MAXWELL'S EQUATIONS AS "EULER-LAGRANGE EQUATIONS" FOR SOME ACTION. THIS REQUIRES INTERPRETING THEM IN ABELIAN GAUGE THEORY ... WHICH CLASSICALLY IS JUST AN OPTION.

SO WE INTRODUCE A CONNECTION A ON A LINE BUNDLE

$\mathcal{L} \to M =$ SPACETIME

AND WE SET

$$F = dA = \text{CURVATURE FORM}$$

THEN OF MAXWELL'S EQUATIONS

$$0 = dF$$
$$0 = d * F$$

WE'VE BROKEN THE SYMMETRY.

THE FIRST IS NOW AN IDENTITY

... THE "BIANCHI IDENTITY"

THE SECOND WILL HAVE TO BE AN EULER-LAGRANGE EQUATION.

FOR THIS, WE NEED AN <u>ACTION</u>

$$I = \frac{1}{4e^2}\int_M F\wedge *F + \underbrace{\frac{i\theta}{8\pi^2}\int F\wedge F}_{\frac{1}{2}\theta\int c_1(\mathcal{L})^2}$$

WHERE ULTIMATELY e BECOMES THE CHARGE OF THE ELECTRON

$(e^2/4\pi\hbar c \simeq 1/137)$

AND θ IS UNFAMILIAR

THE QUANTUM THEORY IS DEFINED BY A PATH INTEGRAL OVER THE SPACE $\mathcal{A}$ OF ALL CONNECTIONS MOD GAUGE TRANSFORMATIONS

$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \; \exp(-I) \qquad (\ldots)$$

SINCE I IS QUADRATIC IN F AND HENCE IN A, THE "PATH INTEGRAL" IS A GIANT GAUSSIAN INTEGRAL ... PLUS LATTICE SUM

A GAUSSIAN INTEGRAL IS TRANSFORMED TO ITSELF BY FOURIER TRANSFORM

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ixy} \exp\left(-\frac{\lambda}{2} x^2\right)$$

$$= \exp\left(-\frac{1}{2\lambda} y^2\right)$$

THERE IS AN ANALOGOUS "POISSON SUMMATION FORMULA" FOR THE SUM OVER CLASSES OF LINE BUNDLE $\mathcal{L}$.

THE COMBINED FOURIER TRANSFORM AND POISSON RESUMMATION MAPS THE ABELIAN GAUGE THEORY TO A THEORY OF THE SAME FORM

BUT WITH

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

REPLACED BY $-\frac{1}{\tau}$

GOING BACK TO THE PATH INTEGRAL

$$\int_{a/g} DA \exp\left(-\frac{1}{4e^2}\int_M F\wedge *F - \frac{i\theta}{2}\int_M c_1(\mathcal{L})^2\right)$$

WE SEE THAT IF M IS SPIN, $\int_M c_1(\mathcal{L})^2 \in 2\mathbb{Z}$

THERE IS A MORE TRIVIAL SYMMETRY $\theta \to \theta + 2\pi$

i.e.

$\tau \to \tau + 1$ $\left(\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right)$

THIS PLUS $\tau \to -1/\tau$ FROM THE FOURIER TRANSFORM COMBINE TO THE GROUP $SL(2,\mathbb{Z})$ OF MATRICES

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ \det M = 1 \end{array}$$

THIS GROUP IS GENERATED BY

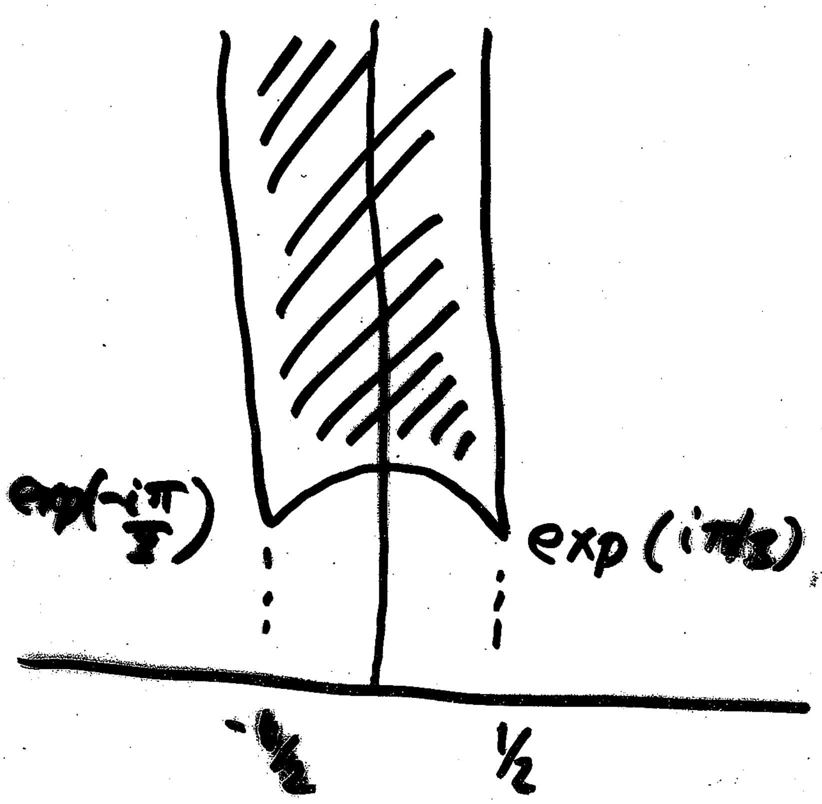
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

AND ACTS ON THE UPPER HALF PLANE BY

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

(IF M IS NOT SPIN, WE MUST CONSIDER THE SUBGROUP GENERATED BY S AND T^2)

SO AS A QUANTUM THEORY, THE FREE $U(1)$ GAUGE THEORY HAS THIS $SL(2,\mathbb{Z})$ SYMMETRY, ACTING ON THE VARIABLE τ WHICH TAKES VALUES IN THE UPPER HALF PLANE.



BY THE ACTION OF $SL(2,\mathbb{Z})$, WE CAN MAP τ TO THE INDICATED REGION

THE SYMMETRY

$S;\ \tau \to -1/\tau$

ACTS BY ELECTRIC-MAGNETIC DUALITY, AS ONE CAN SHOW BY MORE CAREFUL STUDY OF THE FOURIER TRANSFORM...

WE RETURN TO THIS TOMORROW.

THIS IS INTERESTING, BUT IT IS FAR MORE INTERESTING TO FIND A SIMILAR DUALITY IN A NONLINEAR THEORY. SO WE REPLACE THE GAUGE GROUP $U(1)$ BY A COMPACT SIMPLE (OR MORE GENERALLY A COMPACT NONABELIAN) GAUGE GROUP G. WE SET

A = CONNECTION ON G-BUNDLE $E \to M$

$F = dA + A \wedge A$ = CURVATURE

THEN WE FORMALLY PROCEED AS BEFORE

$$I = \frac{1}{4e^2}\int_M \mathrm{Tr}\, F \wedge *F + \frac{i\theta}{8\pi^2}\int_M \mathrm{Tr}\, F \wedge F$$

AND CONSIDER

$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A \; \exp(-I)$$

DOES THIS THEORY HAVE ELECTRIC-MAGNETIC DUALITY?

THE ANSWER TO THAT LAST QUESTION IS "NO." WHAT HAPPENS INSTEAD IS THE SUBJECT OF THE 2004 NOBEL PRIZE, AND OF ONE OF THE CLAY MILLENNIUM PRIZES.

WE DO GET ELECTRIC-MAGNETIC DUALITY IF WE CONSIDER NOT THE "ORDINARY" NONABELIAN GAUGE THEORY, BUT ITS MAXIMALLY SUPERSYMMETRIC EXTENSION "$\mathcal{N}=4$ SUPER YANG-MILLS THEORY"

THIS IS YANG-MILLS THEORY ON A SUPER-MANIFOLD WHOSE BOSONIC REDUCTION IS FOUR-DIMENSIONAL

IT IS HIGHLY EXCEPTIONAL, AS I'LL TRY TO INDICATE IN THIS TABLE:

NUMBER OF SUPERCHARGES		CLASSICAL MODEL	
0		LAPLACIAN ON A MANIFOLD X	
2	$\mathbb{R}$	HODGE THEORY DIFFERENTIAL FORMS ON X	d, d^*
4	$\mathbb{C}$	HODGE THEORY OF A KAHLER MANIFOLD	$\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$
8	$\mathbb{H}$	HODGE THEORY OF A HYPERKAHLER MANIFOLD	
16	$\mathbb{O}$	OCTONIONIC CASE OF HODGE THEORY	

WE REALLY WANT FIELD THEORY, BUT HERE IS THE CLASSICAL MODEL: FIND FIRST ORDER DIFFERENTIAL OPERATORS Q_i, $i = 1 \ldots n$ OBEYING

$$\{Q_i, Q_j\} = 2\delta_{ij} H$$

$H =$ A "LAPLACIAN"

THIS HAS CLASSICAL EXAMPLES FOR $n = 2, 4, 8$, BUT WHAT ABOUT $n = 16$?

FOR EACH G, SUCH A MODEL, WITH $n=16$, ARISES BY "REDUCING" $\mathcal{N}=4$ SUPER YANG-MILLS THEORY

FIELDS A, ϕ, λ AND ACTION

$$I = \frac{1}{4e^2}\int \mathrm{Tr}\, F\wedge *F + \cdots$$

$$+ \frac{i\theta}{8\pi^2}\int \mathrm{Tr}\, F\wedge F$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

SIMPLE G.

THIS THEORY DOES HAVE ELECTRIC-MAGNETIC DUALITY, WITH A "TWIST"

THE SYMMETRY

$$S:\ \tau \to -\frac{1}{n_g\,\tau}$$

MAPS G TO ${}^{L}G$.

ALSO MORE ELEMENTARY

$\tau \to \tau+1 \qquad G \to G$

$n_g = 1, 2,$ OR 3 = RATIO OF LENGTH SQUARED OF LONG & SHORT ROOTS

IN CONTRAST TO THE USUAL LANGLANDS AND GEOMETRIC LANGLANDS STATEMENTS, THE GROUPS G AND ${}^L G$ ENTER IN A COMPLETELY SYMMETRIC WAY. THE SYMMETRY IS BROKEN ONLY BY THE CHOICE OF QUESTION.

NOW $\mathcal{N}=2$ SUPER YANG MILLS

CAN BE "TWISTED" TO MAKE

A TOPOLOGICAL FIELD THEORY

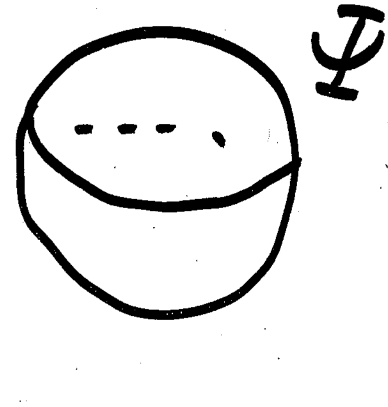
$\Rightarrow$ DONALDSON THEORY

OF FOUR-MANIFOLDS

A SIMILAR PROCEDURE FOR $\mathcal{N}=4$

GIVES A FAMILY OF TFT'S

PARAMETRIZED BY $\mathbb{CP}^1$



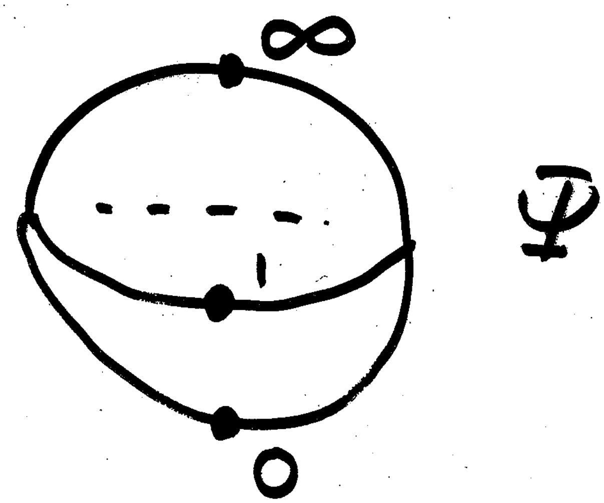
$\Rightarrow$ GEOMETRIC

LANGLANDS

IN FACT THE S-DUALITY

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad \text{ACTS BY}$$

$$\Psi \to \frac{a\Psi+b}{c\Psi+d}$$



IN PARTICULAR, WE GET GEOMETRIC LANGLANDS FROM

$S : \Psi \to -1/n_g \Psi$ ... COMPARE

$\Psi = \infty$ GAUGE GROUP ${}^LG$

TO

$\Psi = 0$ GAUGE GROUP G

NOW TO GO ON, WE SPECIALIZE TO

$$M = \Sigma \times C$$

ANOTHER TWO-MANIFOLD

RIEMANN SURFACE ON WHICH WE'LL DO GEOMETRIC LANGLANDS

IF WE TAKE Σ TO BE MUCH "LARGER" THAN C, THEN THE FOUR-DIMENSIONAL GAUGE THEORY ON M CAN BE "WELL-APPROXIMATED" BY A "SIGMA MODEL" ON Σ

FIELDS VARY SLOWLY ON Σ, AND WHEN RESTRICTED TO $p \times C$ FOR p A POINT IN Σ MUST BE SUCH AS TO MINIMIZE THE ENERGY.

THIS CONDITION GIVES EQUATIONS FIRST STUDIED BY HITCHIN:

FIELDS CONNECTION A ON $E \to C$ AND "HIGGS FIELD"

$$\phi \in \Omega^1(C, ad(E))$$