Birational Geometry in Characteristic $p > 0$

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Algebraic Geometry

- In my previous talk, I discussed the birational geometry of varieties defined over the complex numbers.
- In this talk I will focus on varieties defined over an algebraically closed field of characteristic $p > 0$.
- Throughout this talk $X \subset \mathbb{P}^N_k$ will denote a d-dimensional projective variety defined over an algebraically closed field $k = \bar{k}$ of characteristic $p > 0$.
- Typically we assume that X is smooth or has mild singularities.
- We begin by recalling some of the highlights from the MMP in characteristic 0 that we will use as a guiding principle in characteristic $p > 0$.
- As usual $\omega_X = \wedge^d T^\vee_X$ denotes the **canonical line bundle** so that sections $s \in H^0(\omega_X^{\otimes m})$ can be locally written as $s|_U = f(x_1, \ldots, x_d) dx_1 \wedge \ldots \wedge dx_d$.
- $R(\omega_X) = \oplus_{m \geq 0} H^0(\omega_X^{\otimes m})$ is the **canonical ring**.

The canonical ring

The fundamental result of the MMP in characteristic 0 is

Theorem (Birkar-Cascini-Hacon-McKernan, Siu)

Let X be a smooth projective variety over an algebraically closed field of characteristic 0, then $R(\omega_X)$ is finitely generated.

- Note that $R(\omega_X)$ is a birational invariant.
- The **Kodaira dimension** $\kappa(X) \in \{-1, 0, 1, \ldots, d = \dim X\}$ is given by $\kappa(X) = \mathrm{tr.deg.}_k R(\omega_X) - 1$.
- We say that X is of **general type** if $\kappa(X) = d$. In this case $X_{\mathrm{can}} = \mathrm{Proj}(R(\omega_X))$ is a distinguished representative of the birational class of X with mild singularities such that $\omega_{X_{\mathrm{can}}}^{\otimes m}$ is very ample for some $m > 0$.
- Thus $H^0(\omega_X^{\otimes m})$ defines an embedding $\phi_m : X_{\mathrm{can}} \hookrightarrow \mathbb{P}^n = \mathbb{P}H^0(\omega_X^{\otimes m})$ with $\omega_{X_{\mathrm{can}}}^{\otimes m} = \phi_m^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Geometry of curves

- This allows us to construct projective moduli spaces.
- Eg. When $d = 1$, curves of general type correspond to curves of genus $g \geq 2$ or equivalently such that the degree of the canonical line bundle is positive $\deg(\omega_X) = 2g - 2 > 0$.
- These curves are parametrized by M_g, an irreducible veriety of $3g - 3$ which can be compactified to $M_g \subset \bar{M}_g$ in a geometrically meaningful way.
- The points of $\bar{M}_g \setminus M_g$ correspond to stable curves, i.e. curves with node singularities and ample canonical line bundle.
- It turns out that these results also hold over algebraically closed fields of characteristic $p > 0$.
- In fact there is more good news in dimension $d = 2$

Geometry of surfaces

- When $d = 2$, Bombieri and Mumford (1976) prove that the Enriques classification of surfaces also holds if $char(k) = p > 0$.
- There are a few surprises, such as quasi-hyperelliptic surfaces .
- These have $c_1(\omega_X) \equiv 0$ and are fibered over an elliptic curve, but the fibers are cuspidal rational curves!
- This can not happen in characteristic 0.
- Even more surprisingly Ekhedal (1988) showed that $H^0(\omega_X^{\otimes 5})$ defines an embedding $\phi_5 : X_{\mathrm{can}} \hookrightarrow \mathbb{P}^N$.
- He even shows that $H^i(\omega_{X_{\mathrm{can}}}^{\otimes m}) = 0$ for $i > 0$, $m \geq 2$ except in one case $(p = 2,\ m = 2, \ldots)$.

Geometry of surfaces

Theorem (Alexeev, Kollár, Patakfalvi, Hacon-Kovács and others)

Fix the volume $v = c_1(\omega_{X_{\mathrm{can}}})^2$. Then there exists an integer p_0 such that for all $p > 0$ there is a projective moduli space of stable surfaces $\bar{M}_{2,v}$.

- In fact this moduli space is defined over $\mathbb{Z}[1/m]$.
- It is expected that after some technical issues are resolved, it will be defined over $\mathbb{Z}$.
- The main remaining technical issues are the minimal model program for 3-folds and $p = 2, 3$, inversion of adjunction type results and semistable reduction.
- Next I will discuss some of the technical difficulties that we encounter in positive characteristics.

Technical difficulties

- The first difficulty is that in positive characteristics resolution of singularities is only known in dimensions ≤ 3 (Abhyankar, Cossart-Piltant, Cutkosky and others).
- Conjecturally it is expected in all dimensions and de Jong's theory of alterations provides us with a good substitute.
- A more serious issue is the failure of vanishing theorems.
- Recall **Serre vanishing**: If L is an ample line bundle and F is a coherent sheaf on a projective variety, then $H^i(F \otimes L^{\otimes m}) = 0$ for all $m \gg 0$.
- **Kodaira vanishing** and its generalizations due to Kawamata, Viehweg and others is a much more precise statement: If X is smooth, then $H^i(\omega_X \otimes L) = 0$ for $i > 0$.
- By Kawamata and Viehweg we may even assume that X has some mild (klt) singularities and L is nef and big (instead of ample). This formulation is preferable for birational geometry.
- It is well known that Kodaira vanishing in characteristic $p > 0$ fails as soon as $d \geq 2$ (Raynaud, Lauritzen-Rao and others).

Why is Kodaira vanishing useful?

- Consider $S \subset X$ a smooth divisor in a smooth variety and L an ample line bundle.
- There is a short exact sequence

$$0 \to \omega_X \otimes L \to \omega_X(S) \otimes L \to \omega_S \otimes L|_S \to 0.$$

- Kodaira vanishing implies that $H^0(\omega_X(S) \otimes L) \to H^0(\omega_S \otimes L|_S)$ is surjective.
- Therefore we can deduce results on the geometry of X from results on the geometry of S.
- This allows for proofs by induction on the dimension.
- For example if $S \sim K_X$ is ample and $\omega_S^{\otimes k}$ is base point free, then $\omega_X^{\otimes 2k}$ is base point free.
- Proof: Clearly the base locus is contained in S and we conclude f $\omega_X^{\otimes 2k} \cong (\omega_X(S))^{\otimes k}$ and $H^0((\omega_X(S))^{\otimes k}) \to H^0(\omega_S^{\otimes k})$ is surjective.

Serre vanishing and the Frobenius

- To remedy the failure of Kodaira vanish we combine Serre vanishing with the Frobenius morphism.
- Let $F : X \to X$ be the morphism defined by $F^*(f) = f^p$.
- Note that $(f+g)^p = f^p + g^p$ and $(fg)^p = f^p g^p$ (since $\mathrm{char}(k) = p$) and so we have a ring homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$.
- It is easy to see that if L is a line bundle then $F^*L \cong L^p$ and so by the projection formula $(F^e_*\omega_X) \otimes L \cong F^e_*(\omega_X \otimes L^{p^e})$.
- By Grothendieck duality applied to $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ we have a trace map $tr : F^e_*\omega_X \to \omega_X$ which can be described locally $tr(x^{p(j+1)-1}dx) = x^j dx$.
- The trace map is compatible with adjunction $\omega_X(S) \to \omega_S$.
- Combining this, we have a commutative diagram:

$$\begin{array}{ccc} \omega_X(S)\otimes L & \longrightarrow & \omega_S\otimes L|_S \\ \uparrow{\scriptstyle tr^e} & & \uparrow{\scriptstyle tr^e} \\ F^e_*(\omega_X(S)\otimes L^{p^e}) & \longrightarrow & F^e_*(\omega_S\otimes L^{p^e}|_S) \end{array}$$

$$0 = H^1(\omega_X\otimes L^{p^e}) = H^1(F^e_*(\omega_X\otimes L^{p^e}))$$

by Serre Vanishing

Passing to cohomology

$$\begin{array}{ccc} H^0(\omega_X(S)\otimes L) & \longrightarrow & H^0(\omega_S\otimes L|_S) \\ \uparrow & & \uparrow{\scriptstyle H^0(tr^e)} \\ H^0(\omega_X(S)\otimes L^{p^e}) & \twoheadrightarrow & H^0(\omega_S\otimes L^{p^e}|_S) \end{array}$$

Let $S^0(\omega_S\otimes L|_S) = \mathrm{Im}(H^0(tr^e))$

then

$$H^0(\omega_X(S)\otimes L) \twoheadrightarrow S^0(\omega_X\otimes L|_S)$$

Frobenius stable sectiobs

- The challenge is then to identify the space of **Frobenius stable sections** $S^0(\omega_S \otimes L) \subset H^0(\omega_S \otimes L)$.
- When L is sufficiently ample (and S is smooth), then $S^0(\omega_S \otimes L) = H^0(\omega_S \otimes L)$.
- But when L is "small" this is a subtle problem.
- For example if $L = \mathcal{O}_S$ and S is an elliptic curve, then $S^0(\omega_S) = H^0(\omega_S)$ iff and only if S is ordinary.
- Conjecturally, if S is defined over k of characteristic 0, then $S^0(\omega_{S_p}) = H^0(\omega_{S_p})$ for infinitely many primes.
- Caution: this is not even known for abelian varieties.
- Local versions of this conjecture are also interesting. Eg if S is log canonical, then we expect that S_p is **locally F-split** for infinitely many primes p meaning that $F^e_*\omega_{S_p} \to \omega_{S_p}$ is surjective.

F-regularity

- It is known that if S is klt, then S_p is **strongly F-regular** for all primes $p \gg 0$ (Hara, Mustata-Srinivas, Smith).
- Here, strongly F-regular means that for any effective divisor $D \geq 0$, the induced map $F^e_* \omega_{S_p}(D) \to \omega_{S_p}$ is surjective for $e \gg 0$.
- When $\dim S = 2$ and $p > 5$, Hara shows that klt singularities are exactly the strongly F-regular singularities.
- We were able to leverage this result to prove the existence of flips for 3-folds.

Minimal models of threefolds

Theorem (Hacon-Xu, Birkar, Waldron, Hacon-Witaszek)

Let X be a smooth projective 3-fold over an algebraically closed field of characteristic > 3, then $R(\omega_X)$ is finitely generated and there exists a finite sequence of flips and divisorial contractions to a minimal model $X \dashrightarrow X_{\rm min}$ (so that $X_{\rm min}$ has terminal singularities and $\omega_{X_{\rm min}}$ is nef).

- One of the key steps in the proof is to show the existence of pl-flips (Shokurov).

pl-flips

- Recall that a pl-flip is a flipping contraction $f : X \to \bar{X}$ with $\rho(X/\bar{X}) = 1$, $-K_X - B$ and $-S$ are ample over $\bar{X}$ and $(X, S + B)$ is a plt pair.
- In particular $K_X \sim_{\mathbb{Q}} \lambda S$ for some $\lambda > 0$ (for simplicity we assume $\bar{X}$ is affine and $B = 0$).
- To show the existence of the flip, we must show that $R(K_X)$ is finitely generated (over $\bar{X}$).
- Then the flip $X^+ \to \bar{X}$ is given by $X^+ = \mathrm{Proj}_{\bar{X}}(R(K_X))$.
- This is equivalent to showing that $R(K_X + S)$ is finitely generated.

Finite generation

- From the short exact sequences

$$0 \to (\omega_X(S))^{\otimes m}(-S) \to (\omega_X(S))^{\otimes m} \to\to (\omega_S(B_S))^{\otimes m} \to 0$$

where $K_S + B_S = (K_X + S)|_S$, it follows that $R(K_X(S))$ is finitely generated if so is

$$R_S(K_S + B_S) = \mathrm{Im}(R(K_X + S) \to R(K_S + B_S)).$$

- The rough idea is that the kernel of the above map is a principal ideal defined by the equations of S.
- Note that (S, B_S) is a klt surface and so $R(K_S + B_S)$ is finitely generated and hence the statement would follow if we can show that $S^0(m(K_S + B_S)) = H^0(m(K_S + B_S))$.
- Loosely speaking, we achieve this by applying a generalization of Hara's result.

Finite generation

- Hara's result applies for $p > 5$, however for $p = 5$ we have a detailed description and we can do a case by case analisys.
- The cases $p = 2, 3$ or $d \geq 4$ seem extremely hard and I expect/hope that finite generation of the canonical ring will fail in higher dimensions and low characteristics.