

# Why do matrices commute?

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## ABSTRACT

Put another way: let  $p$  be a polynomial in  $2n^2$  variables that vanishes on each pair of commuting matrices. Say we know that each entry of  $XY - YX$  is small, of order  $\epsilon^2$ . Can we guarantee that  $p(X, Y)$  is also this small? In more flowery language, is the scheme of commuting matrices reduced?

Amazingly, we don't know (and I can't answer the question either). I'll introduce several other related schemes that seem easier to study, like the space of pairs of matrices whose commutator is diagonal. I'll prove that this is a reduced complete intersection, one of whose components is the commuting variety. Surprisingly, it has only one other component – there is “only one way” for two matrices to have a diagonal commutator, other than that they commute – and I'll explain where that one comes from. Along the way we'll also see a rather curious invariant of permutations, and much simple linear algebra.