Mathematics 113 — Professor K. A. Ribet

Another Midterm Exam — September 26, 2013

Morning Edition

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences.* Your explanations are your only representative when your work is being graded.

The problems have equal weight.

1. Find two non-isomorphic finite groups, each with exactly three conjugacy classes. (Explain why the groups have the required property and why they're not isomorphic to each other.)

You can take the cyclic group $\mathbf{Z}/3\mathbf{Z}$ *and the symmetric group* S_3*. They are not isomorphic because one is abelian and the other isn't. Three conjugacy classes? In abelian groups, each element forms its own conjugacy class and* $\mathbf{Z}/3\mathbf{Z}$ *has three elements. In* S_3*, the three 2-cycles form one conjugacy class and the two 3-cycles another; the identity gives you the third conjugacy class.*

2. Suppose that G is a finite group and that N is a normal subgroup of G. Assume that the order of G/N is divisible by p. If P is a p-Sylow subgroup of G, prove that $P/(P \cap N)$ is a p-Sylow subgroup of G/N.

Let p^n *be the largest power of* p *dividing the order of* G *and let* p^a *be the order of* $P \cap N$*. Then* p^{n-a} *is the order of* $P/(P \cap N)$*, which is a subgroup of* G/N*. (If* $\pi : G \to G/N$ *is the canonical map* $g \mapsto gN$*, then* $P/(P \cap N)$ *is nothing other than* $\pi(P)$*. The largest power of* p *dividing the order of* N *is some power* p^b *of* p*, where* $b \geq a$ *because* $P \cap N$ *is a subgroup of* N*. The largest power of* p *dividing the order of* G/N *is then* p^{n-b}*, which is at most* p^{n-a}*. However, we are staring at the subgroup* $P/(P \cap N)$ *of* G/N*, and this subgroup has order* p^{n-a}*. It follows that* $a = b$ *and that* $P/(P \cap N)$ *is indeed a* p*-Sylow subgroup of* G/N*.*

3. Let G be a group (possibly an infinite group), and let $Z(G)$ be the center of G. Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

Let $gZ(G)$ *be a generator of the cyclic group* $G/Z(G)$*. Then each element of the group* G *may be written as a product* $g^i z$ *with* $i \in \mathbf{Z}$ *and* $z \in Z(G)$*. Write down two such products and you'll see that they commute with each other: elements of* $Z(G)$ *commute with everything and powers of* g *commute with each other.*

4. Let H and K be normal subgroups of the group G such that that $H \cap K$ is the trivial group. Show that $hkh^{-1}k^{-1}$ belongs both to H and to K and then prove that $hk = kh$ for all $h \in H, k \in K$.

The product of four terms (which is called a commutator) may be written $(hkh^{-1})k^{-1}$ and also $h(kh^{-1}k^{-1})$. The first expression shows that the product is in K while the second shows it's in H. It's thus in the intersection of the two subgroups and is therefore 1. The equation $hkh^{-1}k^{-1} = 1$ may be rewritten $hk = kh$.

5. Let G be a transitive permutation group acting on the finite set A. We assume that A has at least two elements. As usual, for each $a \in A$ we let G_a be the stabilizer of a in G. Recall (from HW #8) that a *block* is a non-empty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$. Recall also that G is said to be *primitive* if the only blocks are the sets of size 1 and A itself.

a. Prove that if B is a block containing the element a of A, then the subgroup

$$G_B = \{\, \sigma \in G \,|\, \sigma(B) = B \,\}$$

of G contains G_a.

As I'm sure you all recognized, this problem was copied from the book—it's from HW #8, probably §4.1. The book problem asked you to prove that G_B is a subgroup, but I rephrased the problem so that it's already stipulated to be a subgroup. However, we still need to see why it contains G_a. Suppose that σ lies in G_a. Then $\sigma(a) = a$, so that $\sigma(B)$ contains a because a is in B. It follows that B and $\sigma(B)$ are not disjoint; thus they must be equal. Since $\sigma(B) = B$, σ is an element of G_B.

b. Assume that the transitive group G is primitive on A. Prove that, for each $a \in A$, the subgroup G_a of G is maximal (i.e., that there are no subgroups of G containing G_a other than G_a and G).

Since the group acts transitively, all of the stabilizers G_a are conjugate in G. Indeed, if a and b are elements of A, then $b = \sigma(a)$ for some σ. We find then that $G_b = \sigma G_a \sigma^{-1}$, as was discussed in class. Hence one stabilizer is maximal if and only if they all are.

Note also that the stabilizers are proper subgroups of G (i.e., not equal to G). That's because I assumed that A has at least two elements.

Digression: The book doesn't make this assumption, so it's possible in the book for A to be a 1-element set $\{a\}$. Then $G_a = G$. I guess that you'd have to deem G to be a maximal subgroup of G and would have to say that the action is primitive in this case. And what about the case where A is the empty set? There's no reason why a group can't act on $\emptyset$; I'd call the action transitive in this case and would probably even say that it's primitive! These cases are borderline pathological, IMHO.

Now for the proof: Take $a \in A$ and let $H = G_a$. Then we identify A with G/H in the usual way: $gH \in G/H$ is identified with $ga \in A$. Arguing by contradiction, we assume that there is a subgroup K of G containing H with K different from H and different *from* K*. Then* $K/H \subset G/H$ *has more than one element and is not all of* $G/H = A$*. To get a contradiction, it suffices to show that* K/H *is a block.*

For this, we take $g \in G$ *and suppose that* $g(K/H)$ *and* K/H *have an element in common, say* kH*. The aim now is to prove that* $g(K/H) = K/H$*. We will show that* g *lies in* K*, which is enough to prove this equality of sets. We note next that* $g(K/H)$ *is the set of all* gxH *with* x *in* K*. For some* $x \in K$*,* $gxH = kH$*; thus we have* $gx = kh$ *with* x *and* k *in* K *and* $h \in H$*. This implies that* $g = khx^{-1} \in K$*, so we get that* $g(K/H) = K/H$*. Thus* K/H *is indeed a block, which contradicts the assumption that the action is primitive.*