George M. Bergman — Spring 2009, Math H113 — 13 Feb., 2009
61 Evans Hall — Solutions to the **First Midterm** — 3:10-4:00 PM

1. (10 points.) In S_6, let $\sigma = (1\ 2\ 3)(4\ 5)$ and $\tau = (2\ 3\ 4)$. Compute $\tau\sigma\tau^{-1}$, expressing the answer as a product of disjoint cycles. *Answer:* $(1\ 3\ 4)(2\ 5)$.

2. Complete the following definitions. You may use, without defining them, any terms or symbols that our text defines before defining the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in the text, but for full credit they should be clear, and mean the same thing as those definitions.

(a) (5 points) If $f: A \to B$ is a map of sets, and C is a subset of B, then $f^{-1}(C)$ (called the *preimage* of C under f) denotes *Answer:* $\{a \in A \mid f(a) \in C\}$.

(b) (15 points) If G is a group and A is a set, then an *action* of G on A means a function $G \times A \to A$, written $(g,a) \mapsto g \cdot a$, and satisfying *Answer:* (i) *for all* $g, h \in G$ *and* $a \in A$, $g \cdot (h \cdot a) = (g\,h) \cdot a$, *and* (ii) *for all* $a \in A$, $1 \cdot a = a$. Such an action is equivalent to a homomorphism from G to *Answer:* S_A (*the symmetric group on the set* A).

3. (30 points, 10 points each.) For each of the items listed below, either *give an example* with the property stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) Groups G and H, a homomorphism $\varphi: G \to H$, and an element $g \in G$ such that g has finite order, while $\varphi(g)$ has infinite order.

Answer: Does not exist. If $g^n = 1$ *for some* $n \neq 0$, *then* $\varphi(g)^n = 1$, *so* $\varphi(g)$ *has finite order.*

(b) A group G and elements $x \neq y$ in G such that $\langle x \rangle = \langle y \rangle$.

Answer: Many possible examples. E.g., $G = \mathbb{Z}$, $x = 1$, $y = -1$.

(c) A group G and subgroups $H \leq G$, $K \leq G$ such that the set $H \cup K$ is not a subgroup of G.

Answer: Many possible examples. E.g., $G = \mathbb{Z}$, $H = \langle 2 \rangle$, $K = \langle 3 \rangle$.

4. (20 points) If $f: X \to Y$ is any function between sets, let the *graph* of f mean the subset of $X \times Y$ consisting of all pairs of the form $(x, f(x))$ with $x \in X$.

Suppose G and H are groups. We recall that the set $G \times H$ is made a group by defining $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.

Show that if $f: G \to H$ is a function such that the graph of f is a subgroup of $G \times H$, then f is a homomorphism from G to H.

You may assume any results proved in our readings so far. You may write $\mathrm{graph}(f)$ for the graph of the function f.

Answer: We must show that for any $x, y \in G$ *one has* $f(xy) = f(x)f(y)$. *Now by definition,* $\mathrm{graph}(f)$ *contains the elements* $(x, f(x))$ *and* $(y, f(y))$. *Hence, being a subgroup, it contains their product* $(xy, f(x)f(y))$. *By the definition of* $\mathrm{graph}(f)$, *this means that* $f(xy) = f(x)f(y)$, *as required.*

5. (20 points) Suppose G is a nontrivial group (a group with more than one element), written multiplicatively, which has no subgroups other than $\{1\}$ and G itself. Prove that G is a finite cyclic group whose order is a prime. You may assume any results proved in our readings so far.

Answer: Since G *is nontrivial, we can take an element* $x \neq 1$ *in* G. *Then* $\langle x \rangle$ *is a subgroup of* G *not equal to* $\{1\}$, *so by assumption,* $\langle x \rangle = G$; *so* G *is cyclic.*

If x *had infinite order, then by a result in the book, the only generators of* $\langle x \rangle$ *would be* $x^{\pm 1}$, *so nonidentity elements not of this form would generate other nontrivial subgroups, e.g.,* $\langle x^2 \rangle$. *So* x *has finite order, so* $G = \langle x \rangle$ *is a finite group.*

If x *had composite order* $|x| = mn$, *where* m *and* n *are both* > 1, *then* x^m *would have order* n, *so* $\langle x^m \rangle$ *would have order* n, *which is* > 1 *but* $< mn$, *making it a nontrivial proper subgroup. So this also cannot happen, so* p *is prime.*

☛ Reminder: The reading for Wednesday, February 18, is #8 ☚