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Spring 2006, Math 104
Solutions to the **Second Midterm**

24 March, 2006
3:10-4:00 PM

1. (32 points, 8 points each.) Complete the following definitions. You may use, without defining them, terms or symbols that Rudin defines before he defines the word or symbol asked for. Your definitions do not have to have exactly the same wording as those in Rudin, but for full credit they should be clear, and mean the same thing as his.

(a) A subset E of a metric space X is said to be *compact* if

Answer: *Every open covering of* E *has a finite subcovering.*

(b) A series $\Sigma_{n=1}^{\infty} a_n$ is called *absolutely convergent* if *Answer:* $\Sigma_{n=1}^{\infty} |a_n|$ *is convergent.*

(c) If f is a function from a segment $(a, b) \subseteq R$ to a metric space X, and if $u \in (a, b)$ and $x \in X$, then we write $f(u+) = x$ if

Answer: $\lim_{n\to\infty} f(t_n) = x$ *for every sequence* (t_n) *in* (u, b) *such that* $\lim_{n\to\infty} t_n = u$. *(It's OK, of course, to use "$\to$" notation, as Rudin happens to do in this definition, instead of the "lim" notation that I used above.)*

(d) If f is a bounded function on an interval $[a, b]$, then $\bar{\int} f dx$ denotes $\inf U(P, f)$. Here the inf is taken over all *Answer: partitions* P *of* $[a, b]$, and $U(P,f)$ denotes $\Sigma_{i=1}^{n} M_i \Delta x_i$, where for each i, $M_i =$ *Answer:* $\sup f(x)$ $(x \in [x_{i-1}, x_i])$ and $\Delta x_i =$ *Answer:* $x_i - x_{i-1}$.

2. (32 points, 8 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) A finite subcovering of the covering $\{(x-0.01, x+0.01) \mid x \in [0,1]\}$ of the interval $[0, 1]$.

Answer: $\{(0.01(n-1), 0.01(n+1)) \mid n = 0, 1, \ldots, 100\}$ *is the simplest example.*

(b) A sequence (s_n) of points of the segment $(0,1)$, such that $\lim_{n\to\infty} s_n$ exists in R, but does not belong to $(0, 1)$. *Answer: Easy examples are* $s_n = 1/(n+1)$ *and* $s_n = 1 - 1/(n+1)$.

(c) A metric space X, and a Cauchy sequence in X which does not converge in X.

Answer: Many examples; e.g., the same example as for (b), *with* $X = (0, 1)$.

(d) A differentiable function $f: (-1, 1) \to R$ such that $f'(x) \leq -1$ for all $x < 0$, and $f'(x) \geq 1$ for all $x > 0$.

Answer: Does not exist. The derivative of a function on an interval takes on all values between the values at the endpoints, so if we had such a function f, $f'([-½, ½])$ *would include the infinitely many values between* $f'(-½) \leq -1$ *and* $f'(½) \geq 1$. *But by assumption,* f' *takes on no value in* $(-1, 1)$ *except perhaps* $f'(0)$.

3. (18 points) If X, Y and Z are metric spaces, and if $f: X \to Y$ and $g: Y \to Z$ are uniformly continuous functions, and we define $h: X \to Z$ by $h(x) = g(f(x))$, show that the function h is also uniformly continuous.

Answer: Given $\varepsilon > 0$, *uniform continuity of* g *implies that we can find* $\delta_1 > 0$ *such that for all* $y, y' \in Y$ *with* $d(y, y') < \delta_1$ *one has* $d(g(y), g(y')) < \varepsilon$. *We can now use uniform continuity of* f *(with* δ_1 *in the role of "*ε*") to get a* δ *such that for any* $x, x' \in X$ *with* $d(x, x') < \delta$, *one has* $d(f(x), f(x')) < \delta_1$.

Hence for any $x, x' \in X$ *with* $d(x, x') < \delta$, *if we put* $f(x)$ *and* $f(x')$ *in the roles of* y *and* y' *in the first sentence, we get* $d(g(f(x)), g(f(x'))) < \varepsilon$, *i.e.,* $d(h(x), h(x')) < \varepsilon$, *which is the condition needed to establish uniform continuity of* h.

4. (18 points) Suppose f is a differentiable function on a segment (a, b) $(a, b \in R)$, and is unbounded. (I.e., there is no $M > 0$ such that $|f(x)| < M$ for all $x \in (a, b)$.) Prove that f' is also unbounded.

Answer: Given $M > 0$, *by unboundedness of* f *there exist* c *and* d *in* (a, b) *such that* $|f(d) - f(c)| \geq M(b-a)$. *We may assume without loss of generality that* $c < d$. *The Mean Value Theorem then tells us that there will exist* $x \in (c, d)$ *such that* $f'(x)(d-c) = (f(d) - f(c))$. *Combining with the preceding inequality, we see that* $|f'(x)|(d-c) \geq M(b-a) > M(d-c)$, *so dividing by* $d-c$, *we get* $|f'(x)| > M$, *as needed to show* f *unbounded.*

(One can also do the proof by contradiction, proving boundedness of f *assuming* f' *bounded, again by the Mean Value Theorem.)*

☛ Reminder: The reading for Monday, April 3 is #24 ☚