Math 110–1, Linear Algebra
Spring 2000, Instructor: I. Novik

Final Exam

Name:____

SID:____________________

This exam is closed book, no notes are allowed. You have 3 hours to complete it.

GOOD LUCK!

1

a	
b	
c	
d	

2

a	
b	
c	
d	
e	
f	
g	
h	

3

a	
b	
c	

4

a	
b	
c	

1. **(16 pts)** This part consists of 4 questions. Each question is worth **4** pts. In each question give an example with required properties. You do NOT have to explain your example.

   (a) A vector space V with two subspaces S and T such that $\dim S = \dim T = 4$, but $\dim(S+T) = 6$.

   (b) A function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $f(ax) = af(x)$ for all $a \in \mathbf{R}$ and all $x \in \mathbf{R}^2$ but f is not linear.

   (c) A non-diagonalizable matrix whose characteristic polynomial is
   $$(3-t)^3.$$

   (d) A normal operator on $\mathbf{R}^2$ that is not diagonalizable.

2. **(49 pts)** Supply a short proof for each of the following statements.

   (a) **(4 pts)** If S, T are invertible linear transformations that commute, then their inverses also commute.

   (b) **(4 pts)** If $T : V \to V$ is unitary and self-adjoint, then $T^2 = I$.

   (c) **(4 pts)** If ϕ is a non-zero functional on an n-dimensional vector space, then the nullity of ϕ equals $n - 1$.

   (d) **(7 pts)** If $T : V \to V$ has the property that T^2 possesses a nonnegative eigenvalue λ^2, then at least one of λ or $-\lambda$ is an eigenvalue for T.
   (Hint: $T^2 - \lambda^2 I = (T + \lambda I)(T - \lambda I)$).

(e) **(7 pts)** Let $V = \mathbf{R}^+$ be the set of positive real numbers. Define the "sum" of two elements x, y in V to be their product xy (in the usual sense), and define "multiplication of an element x in V by a scalar c" to be x^c. Prove that V is a vector space over $\mathbf{R}$ with 1 as the zero element.

(f) **(7 pts)** Prove that if W_1 and W_2 are subspaces of a finite-dimensional real inner-product space V such that $\dim W_1 = \dim W_2$, then there exists an orthogonal transformation T such that $T(W_1) = W_2$.

$A = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

(g) **(7 pts)** If A is a non-zero, strictly upper-triangular $n \times n$ matrix (that is, $A_{ij} = 0$ for all $1 \leq i \leq j \leq n$), then A is not diagonalizable. (Hint: what are the eigenvalues of A?)

(h) **(9 pts)** Let $P_m(\mathbf{R})$ be the vector space consisting of all polynomials of degree $\leq m$ with real coefficients. Suppose that $p_0, p_1, \ldots, p_m$ are elements of $P_m(\mathbf{R})$ with the property that $p_j(1) = 0$ for all j. Prove that the set $\{p_0, p_1, \ldots, p_m\}$ is linearly dependent in $P_m(\mathbf{R})$.
(Hint: Do $p_0, p_1, \ldots, p_m$ generate $P_m(\mathbf{R})$?)

3. **(18 pts)** Let V be a finite-dimensional vector space. Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V. Let $\phi_1, \ldots, \phi_n \in V^*$ be the linear functionals defined by

$$\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

   (a) **(7 pts)** Prove that for every $f \in V^*$, $f = \sum_{j=1}^{n} f(v_j)\phi_j$.

   (b) **(5 pts)** Conclude that $\beta^* = \{\phi_1, \ldots, \phi_n\}$ is a basis for V^*, called the *dual basis* of β.

   (c) **(6 pts)** Let $V = M_{2\times 2}(\mathbf{R})$, let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

   be a basis of V, and let $\beta^* = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ be the dual basis. Show that $f(A) = \operatorname{tr}(A)$ is a linear functional on V, and compute the coordinate vector of this linear functional with respect to β^*.

4. **(17 pts)** Let V be the set of all real sequences $(x_n)_{n=1}^{\infty}$. Define the sum of two such sequences and the multiplication by a scalar $c \in \mathbf{R}$ by

$$(x_n) + (y_n) = (x_n + y_n) \text{ and } c \cdot (x_n) = (cx_n),$$

respectively.

(a) **(5 pts)** Prove that V is a vector space over $\mathbf{R}$.

(b) **(4 pts)** Define $T : V \to V$ as follows: for $x = (x_n)_{n=1}^{\infty} \in V$, let $T(x) = (a - x_n)_{n=1}^{\infty}$, where $a = \frac{1}{100} \sum_{i=1}^{100} x_i$. Prove that T is linear.

(c) **(8 pts)** Prove that T has only two eigenvalues $\lambda = 0$ and $\lambda = -1$ and determine the eigenvectors belonging to each such λ.