MATH 126: FINAL, SPRING, 2000

There are six problems in the exam. Do *all* of them.
Total score: 175 points.

Problem 1.

(i) (10 points) *Find the general solution of the PDE*

$$u_x - 2yu_y = 0.$$

on $\mathbb{R}_x \times \mathbb{R}_y$.

(ii) (10 points) *Now impose in addition that* $u(0,y) = y^2$. *Find* u *explicitly.*

(iii) (10 points) *Consider the PDE*

$$u_x - 2yu_y + u = 0$$

on $\mathbb{R}_x \times \mathbb{R}_y$. *Find its general solution.*

Problem 2. *Consider the function* $f(x) = x$ *on the interval* $[0, \pi]$.

(i) (10 points) *Find the coefficients of the Fourier cosine series,* $A_0/2 + \sum_{n=1}^{\infty} A_n \cos nx$, *of* f.

(ii) (7 points) *Show explicitly that this Fourier series converges uniformly on* $[0, \pi]$.

(iii) (7 points) *Which function does this Fourier cosine series represent outside the interval* $[0, \pi]$? *Sketch its graph on* $[-2\pi, 2\pi]$.

(iv) (6 points) *We wish to approximate* f *by a function* g *of the form* $a_0 + a_2 \cos 2x$ *on* $[0, \pi]$. *Find the constants* a_0 *and* a_2 *that minimize the* L^2 *error of the approximation.*

Problem 3. (25 points) *For both of the following functions* f *on* $[0, l]$, *state whether the Fourier cosine series on* $[0, l]$ *converges in each of the following senses: uniformly, pointwise, in* L^2. *If the Fourier series converges pointwise, state what it converges to for each* $x \in [0, l]$. *Make sure that you give the reasoning that led you to the conclusions.*

(i) $f(x) = x(\sin(\pi x/l))^2$,

(ii) $f(x) = 0$, *for* $0 \leq x \leq l/2$, *and* $f(x) = 1$ *for* $l/2 < x \leq l$.

Problem 4. *Let* D *be a bounded region (open set) in* $\mathbb{R}^2$, *and let* h *be a given continuous function on the boundary of* D. *We wish to solve the Dirichlet problem for the Laplacian, i.e. we wish to find a function* $u \in C^2(D) \cap C^0(\bar{D})$ *that is harmonic on* D, *i.e.* $\Delta u = 0$ *on* D, *and* u *is equal to the given function* h *on the boundary of* D.

(i) (8 points) *State the maximum principle for harmonic functions.*

(ii) (8 points) *Is the solution of this problem unique (if it exists)? Explain briefly why.*

(iii) (7 points) *If* u *solves this problem, can you say for sure that* u *is smoother than* C^2 *in the open set* D? *Why?*

(iv) (7 points) *Suppose* $h \geq 0$ *on the boundary of* D, *and* u *solves this problem. Show that* $u \geq 0$ *on* $\bar{D}$.

Problem 5. *In this problem we consider a square plate of size a with three sides kept at temperature 0, and the fourth at a specified temperature. We wish to find the steady state temperature $u = u(x,y)$ of the plate. That is, we wish to solve $u_{xx} + u_{yy} = 0$ on $D = (0,a)_x \times (0,a)_y$, $u \in C^2(D) \cap C^0(\bar{D})$, with boundary conditions $u(x,0) = 0$, $u(x,a) = 0$ for $0 \le x \le a$, $u(0,y) = 0$, $u(a,y) = g(y)$, $0 \le y \le a$, where g is a given continuous function, $g(0) = 0 = g(a)$.*

(i) (15 points) *Using separation of variables, show that the general solution of the PDE with the homogeneous boundary conditions is*

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh(n\pi x/a) \sin(n\pi y/a).$$

(ii) (10 points) *Find A_n in terms of g.*

(iii) (5 points) *If $g(y) = \sin(2\pi y/a)$, find u explicitly.*

Problem 6. *We want to solve the forced wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$ on the interval $[0,l]$ with homogeneous Dirichlet boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, and initial conditions $u(x,0) = 0$, $u_t(x,0) = 0$. Below you may assume that f is continuous, $f(0,t) = f(l,t) = 0$ for all t. Recall that the solution of the forced wave equation $v_{tt} - c^2 v_{xx} = F(x,t)$ on $\mathbb{R}_x \times \mathbb{R}_t$ with vanishing initial conditions is given by*

$$v(x,t) = \frac{1}{2c} \int_{\Delta} F = \frac{1}{2c} \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} F(y,s)\, dy \right) ds;$$

here Δ is the backward characteristic triangle from (x,t).

(i) (10 points) *Which is the appropriate extension of f to $\mathbb{R}_x \times \mathbb{R}_t$ that reduces the solution of the original problem to that of a problem on the whole real line? Write down the solution of the original problem.*

(ii) (10 points) *Suppose that $|f(x,t)| \le M$ for all $(x,t) \in [0,l] \times \mathbb{R}$. Show that $|u(x,t)| \le Mt^2/2$.*

(iii) (10 points) *Find a constant $C > 0$ such that $|u(x,t)| \le Ct$ for all $t > 0$.*