Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2022

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1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.
2. The exam lasts 3 hours each day, including time to enter questions in gradescope.
3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.
4. Submit your answers is by taking pictures of them and uploading them to gradescope.
5. The exam is open book: you may use notes or books or calculators or the internet, but should not consult anyone else.
6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee If a correction or announcement is needed during the exam it will be sent as an email to the address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Suppose that x, y, p, q are real numbers with $x \geq 0$, $y \geq 0$, $p > 1$, $1/p + 1/q = 1$. Prove Young's inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Solution:

YOUR EXAM NUMBER ___

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Problem 2A. *Score:*

Suppose $f : [0,1] \to R$ is a continuous function with

$$\int_0^1 x^n f(x) dx = 0$$

for all integers n with $1 \leq n < \infty$. Prove that f is identically 0.

Solution:

YOUR EXAM NUMBER ___

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Problem 3A. *Score:*

Suppose that X is an uncountable subset of the reals. Prove that there is a point of X that is a limit of a sequence of distinct points of X.

Solution:

YOUR EXAM NUMBER ____

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Problem 4A. *Score:*

(a). Show that there is a function $f(z)$, holomorphic (analytic) near $z = 0$, such that

$$f(z)^{10} = \frac{1}{\cos(z^5 + 2z^7)} - 1$$

for all z in a neighborhood of $z = 0$.

(b). Find the radius of convergence of its power series about $z = 0$. Your answer may involve a root of an explicitly given polynomial.

Solution:

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Problem 5A. *Score:*

Let $f(z)$ be a function holomorphic on the whole complex plane $\mathbb{C}$ such that $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$. Show that $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$.

Solution:

YOUR EXAM NUMBER ____

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Problem 6A. *Score:*

Let T be a linear transformation of a vector space V into itself. Suppose that $T^{m+1} = 0$, $T^m \neq 0$ for some positive integer m. Show that there is a vector x such that x, $Tx, \ldots, T^m x$ are linearly independent.

Solution:

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Problem 7A. *Score:*

Suppose n is a positive integer and let f be the function $f(x) = (1, x, x^2, ...x^{n-1})$ from $\mathbb{R}$ to $\mathbb{R}^n$, Show that a hyperplane (of codimension 1) containing the points $f(1), f(2), ..., f(n)$ does not pass through the origin.

Solution:

YOUR EXAM NUMBER ___

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Problem 8A. *Score:*

Let A be an abelian group. Suppose that $a \in A$ and $b \in A$ have orders h and k, respectively, and that h and k are relatively prime.

Let r and s be integers. Show that if $ra = sb$ then $ra = sb = 0$.

Solution:

YOUR EXAM NUMBER ____

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Problem 9A. *Score:*

Let $\mathbf{F}$ be a field and let X be a finite set. Let $R(X, \mathbf{F})$ be the ring of all functions from X to $\mathbf{F}$, endowed with the pointwise operations. What are the maximal ideals of $R(X, \mathbf{F})$?

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2022

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YOUR EXAM NUMBER ___

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Problem 1B. *Score:*

Which of the following series converge? Give reasons.

1. $$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}\,.$$

2. $$\sum_{n=2}^{\infty} \frac{1}{n^{1+1/(\log n)^2}}\,.$$

Solution:

YOUR EXAM NUMBER ____

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Problem 2B. *Score:*

Suppose that f is a smooth function from the reals to the reals satisfying the differential equation

$$f'(x) = \sin(f(x))e^{-x^2}$$

Prove that f is bounded.

Solution:

YOUR EXAM NUMBER ____
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Problem 3B. *Score:*

Let the function f be given by $f(x) = 0$ if x is irrational and $f(x) = 1/n^2$ if $x = m/n$ where m, n are coprime integers and $n > 0$. Show that there is a point where f is continuous but not differentiable.

Solution:

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Problem 4B. *Score:*

Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\, dx \ .$$

Solution:

YOUR EXAM NUMBER ____

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Problem 5B. *Score:*

Suppose that the complex function f is holomorphic and bounded for $\Re(z) > 0$. Prove that it is uniformly continuous for $\Re(z) > 1$.

Solution:

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Problem 6B. *Score:*

Prove that a complex square matrix of finite order is diagonalizable. Give an example of a square matrix of finite order (over some other algebraically closed field) that is not diagonalizable.

Solution:

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Problem 7B. *Score:*

Find the number of conjugacy classes of complex 5 by 5 matrices such that all eigenvalues are 1.

Solution:

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Problem 8B. *Score:*

Let G be a finite group and H be a subgroup.

(a) Show that the number of subgroups of G conjugate to H divides the index of H.

(b) Show that if

$$G = \bigcup_{g \in G} gHg^{-1}$$

then $G = H$.

Solution:

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Problem 9B. *Score:*

Let $p(z)$ be a polynomial with real coefficients such that $p(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Show that if the degree of $p(z)$ is d then $d!p(z) \in \mathbb{Z}[z]$.

Solution: