Math 113: Introduction to abstract algebra.

H. W. Lenstra, Jr. (879 Evans, tel. 643–7857, e-mail hwl@math).

Fall 1995

Final examination, Monday, December 11, 12:30–3:30 p.m., 60 Evans.

Name:

Note. You have to do **four** out of the five problems. Cross out the problem that you don't want to be graded. *Give complete proofs of the assertions you are making and of the correctness of your answers.* Theorems proved in the book or in class may be used without proof (but do give the formulation).

1	
2	
3	
4	
5	
Total	

Problem 1.

Let R be a commutative ring, and let A be the subset

$$A = \{e \in R \mid e^2 = e\}$$

of R.

(a) Prove: for all a, $b \in A$ one has $a + b - 2ab \in A$ (here $2ab$ denotes the element $ab + ab$ of R).

(b) Prove that A is an abelian group with the operation $\oplus$ defined by $a \oplus b = a + b - 2ab$ (for a, $b \in A$), and that every non-zero element of that abelian group has order 2.

Solution:

Problem 2.

Prove that one has $a^{13} \equiv a \bmod 2730$ for every $a \in \mathbb{Z}$.

Solution:

Problem 3.
Let T be a set, and let $P(T)$ be the set of all subsets of T. You know from class that $P(T)$ is a ring with respect to the operations $+$ and $\cdot$ defined by

$$A + B = (A \cup B) - (A \cap B), \qquad A \cdot B = A \cap B,$$

for A, $B \subseteq T$. Fix a subset $U \subset T$, and define $f: P(T) \to P(U)$ by

$$f(A) = A \cap U.$$

Prove that f is a ring homomorphism. Prove also that the kernel of f is a *principal* ideal of $P(T)$.

Solution:

Problem 4.

Let α be an element of an extension field of $\mathbb{Z}_2$ with the property that $\alpha^3 = \alpha + 1$.

(a) How many elements does the field $\mathbb{Z}_2(\alpha)$ have?

(b) Find the order of α in the multiplicative group $\mathbb{Z}_2(\alpha)^*$.

Solution:

Problem 5.

Let α denote the real number $\sqrt{5+\sqrt{5}}$.

(a) Find the irreducible polynomial of α over $\mathbb{Q}$.

(b) Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

(c) Find the irreducible polynomial of α over $\mathbb{Q}(\sqrt{5})$.

Solution: