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100 Lewis Hall

Fall 2005, Math 55
Second Midterm

31 October, 2005
1:10-2:00

1. (24 points, 8 points each.) Short answer questions. A correct answer will get full credit whether or not work is shown. An incorrect answer may get partial credit if work is given that follows a basically correct method.

(a) How many 8-element subsets does a 100-element set have? You may express your answer in any of the forms used in the text.

(b) In how many ways can one put together a packet of 30 pieces of fruit, if one has 5 kinds of fruit in unlimited supply, and all that matters is how many of each kind go into the packet? Again, you may express your answer in any of the forms used in the text.

(c) What is the probability that an integer chosen at random from $\{0, 1, \ldots, 10\}$ (where all members of this set have equal probability of being chosen) is odd? (Note: Do not confuse $\{0, 1, \ldots, 10\}$ with $\{1, \ldots, 10\}$.)

2. (24 points, 8 points each.) Complete the following definitions. Your definitions do not have to have the same wording as those in the text, but for full credit they should be clear, and be equivalent in meaning to those.

(a) Integers m and n are said to be *relatively prime* if . . .

(b) A set S is said to be *countable* if . . .

(c) The *lexicographic order* on the set of length-n strings of integers is defined by considering a string $(a_1, \ldots, a_n)$ to precede a string $(b_1, \ldots, b_n)$ under this order if . . .

3. (24 points, 8 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, you should name a particular one. If you give a reason why no example exists, don't worry about giving reasons for your reasons; a simple statement will suffice.

(a) Two integers a and b, neither of which is divisible by 17, such that $a^{16} \not\equiv b^{16} \pmod{17}$.

(b) A program which never halts. (Use pseudocode if you give an example.)

(c) A one-to-one function from the set of ordered pairs (a, b) with $a, b \in \{1, \ldots, 5\}$ to the set of ordered pairs (c, d) with $c \in \{1, \ldots, 6\}$, $d \in \{1, \ldots, 4\}$.

4. (28 points, 14 points each.) Short proofs. I am giving you a page for each, in case some of you give roundabout proofs or have several false starts. But concise proofs should take less than half a page each.

(a) Show that if a, b, A, B are integers and m a positive integer, with $a \equiv A \pmod{m}$ and $b \equiv B \pmod{m}$, then $ab \equiv AB \pmod{m}$. Since this is a result proved in Rosen (though with slightly different notation), you may not call on that result, or results proved from it, in your proof of this statement.

(b) Prove that for every nonnegative integer n, one has $\Sigma_{i=0}^{n} i\, 2^i = (n-1)\, 2^{n+1} + 2$.