Final Exam/math110/fall 2003 LIU
Total Time: 12:30am-3:30pm
Total Score: 200 points

1. (15%) Please determine if the following statements are true or not. Give a brief reasoning for each of your answers.

(a). All the unitary operators on a finite dimensional vector space over $\mathbf{C}$ are normal. (3%)

(b). All $n \times n$ matrices in $\mathbf{M}_{n\times n}(\mathbf{R})$ have their associated Jordan canonical forms over $\mathbf{R}$. (3%)

(c). Let V be a finite dimensional inner product space and $T \in \mathcal{L}(V)$. Then $R(T^*) = N(T)$. (3%)

(d). Let V be a finite dimensional inner product space. An operator $T \in \mathcal{L}(V)$ is self-adjoint iff $[T]_\beta = [T]^*_\beta$ for all the ordered bases β. (3%)

(e). All the orthogonal transformations on a finite dimensional vector space over $\mathbf{R}$ are onto. (3%)

2. (20%) (a). Prove Schur's theorem, I.e. when the characteristic polynomial of a linear transformation $T \in \mathcal{L}(V)$ (on a finite dimensional inner product vector space V over $F = \mathbf{R}$ or $\mathbf{C}$) splits, then there exists an orthonormal basis β such that $[T]_\beta$ is upper-triangular. (14%)

(b). Prove that a self-adjoint operator T over $\mathbf{C}$ must be diagonalizable by an orthonormal basis. I.e. $\exists$ an orthonormal basis β such that $[T]_\beta$ is diagonal. (6%)

3.(25%) (a). Consider the matrix $A = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$.

Show that A is normal and diagonalize A by an orthogonal matrix. (10%)

(b). For the same A, write $L_A = \lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3$ and find T_1, T_2, T_3 explicitly. (5%)

(c). Let $T : P_4(\mathbf{R}) \mapsto P_4(\mathbf{R})$ be $T(f) = f'' + f$. Determine the dot diagram and write down the Jordan canonical form of T. (10%)

4. (20%) (a). Let U be an unitary operator upon an inner product space $(V, <,>)$ over $\mathbf{R}$, i.e. $||U(x)|| = ||x||$ for all $x \in V$.

Prove that $< U(x), U(y) >=< x, y >$ for all x, y. (6%)

(b). Suppose that $T_1, T_2 \in \mathcal{L}(V)$ are linear operators over an inner product space $(V, <,>)$ such that the identity $< x, T_1(y) >=< x, T_2(y) >$ holds for all $x, y \in V$. Show that $T_1 = T_2$. (5%)

(c). Let W be a finite dimensional subspace of the inner product space $(V, <,>)$. Prove that an arbitrary vector $x \in V$ can be decomposed uniquely into the form $x = u + z$, where $u \in W$ and $z \in W^\perp$. (9%)

5. (20%) (a). Prove that the eigenvalues of a self-adjoint operator are all real. (7%)

(b). Show that all eigenvalues of anti-self-adjoint $T^* = -T$ operators are purely imaginary (i.e. $= \sqrt{-1}r, r \in \mathbf{R}$). (4%)

(c). Determine all the operators $T \in \mathcal{L}(V)$ with $T^3 = T$, $T^* = -T$. What can T be? Write down your argument. (9%)

6. (25%) Let $(V, <,>)$ be a finite dimensional inner product vector space over $\mathbf{R}$.

(a). Prove that a linear functional $f \in \mathcal{L}(V, \mathbf{R})$ can always be written as $f(x) =< x, \mathbf{v} >$ for some $\mathbf{v} \in V$. (14%)

(b). Let $\beta = \{v_1, v_2, \cdots, v_n\}$ be an orthonormal basis of $(V, <,>)$. Prove that $< x, y >= [x]^t_\beta [y]_\beta$ for all $x, y \in V$. (6%)

For an $x \in V$, $[x]_\beta$ means the column vector of coordinates relative to β.

(c). Let $T : \mathbf{R}^4 \mapsto \mathbf{R}^4$ be defined by $T(a, b, c, d) = (a + b, c + d, a - c, a + b + c + d)$. Please find $T^* : \mathbf{R}^4 \mapsto \mathbf{R}^4$ explicitly. (5%)

7. (15%) (a). Let W_1, W_2 be two finite dimensional vector subspaces of V. Prove that $dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2)$. (9%)

(b). Let β_1, β_2 be bases of W_1 and W_2, respectively. Show that when $W_1 \cap W_2 = \{0\}$, $\beta_1 \cup \beta_2$ is a basis of $W_1 + W_2$. (6%)

8. (15%) Let V be a finite dimensional vector space over $\mathbf{R}$.

(a). Prove that when W is a T invariant subspace, extend a basis γ of W to a basis β of V. Prove that $[T]_\beta$ is of the following form, (7%)

$$\begin{pmatrix} [T]_\gamma & B \\ \mathbf{0} & C \end{pmatrix}.$$

(b).Let W be a T-invariant sub-space of V. Prove that the characteristic polynomial of T_W divides the characteristic polynomial of $T \in \mathcal{L}(V)$. (8%)

9. (15%) (a). Prove that for $A \in \mathbf{M}_{n\times n}(\mathbf{R})$, $dim_{\mathbf{R}} span(\{I, A, A^2, \cdots\}) \leq n$. (9%)

(b). Give an $n \times n$ example that $dim_{\mathbf{R}} span(\{I, A, A^2, \cdots\}) = n$. (6%)

10. (15%) Prove the following statement: Let V and W be finite dimensional vector spaces having ordered bases β and γ, respectively and let $T \in \mathcal{L}(V, W)$. Then for all $u \in V$, we have $[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.

11. (15%) (a). Prove that two finite dimensional vector spaces V and W are isomorphic to each other **if and only if** $dim(V) = dim(W)$. (10%)

(b). Show that a linear transformation $T \in \mathcal{L}(V, W)$ cannot be onto if $dim(V) < dim(W)$. (5%)