Math 113, Introduction to Abstract Algebra (Kedlaya, fall 2002)
First midterm exam, Wednesday, October 2, 2002

This text is closed-book. No notes or other aids are permitted. Please write all answers on the exam itself, not in a blue book.

Problem 1. Give a one-sentence answer to each of the following questions. (5 points each)

(a) Give a definition of an isomorphism between a group G with operation $*$ and another group G' with operation $*'$.

(b) State the result of the division algorithm, for n an integer being divided by a positive integer m.

(c) Give a definition of the symmetric group S_n. (You need to specify the elements of the group and the group operation, but you do not need to verify the group axioms.)

(d) Give a definition of a left coset of a subgroup H of a group G.

Problem 2.

(a) How many generators does $\mathbb{Z}_{49}$ have? You may give an explanation of where you got your answer in place of listing the generators explicitly. (5 points)

(b) Determine the subgroup of $\mathbb{Z}_{24}$ generated by 8 and 12. (5 points)

(c) Find the order of 531 in the group $\mathbb{Z}_{1065}$. (Hint: find a smaller generator of the same cyclic subgroup.) (5 points)

Problem 3. Let σ and τ be the following permutations of $\{1, 2, \ldots, 8\}$:

$$\sigma = (1, 5, 2, 3)(4, 7), \qquad \tau = (1, 8, 3, 4, 6, 2)(5, 7).$$

(a) Compute $\sigma\tau$. (5 points)

(b) What are the orders of σ and τ? (5 points)

(c) Write τ as a product of transpositions, and determine whether τ is even or odd. (10 points)

Problem 4.

(a) List all possible abelian groups of order 36, up to isomorphism. (5 points)

(b) List the left cosets of the subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4$ generated by $(1, 2)$. (Hint: first list the elements of the subgroup.) (10 points)

Problem 5. Give careful proofs of the following statements. (10 points each)

(a) The intersection of two subgroups H_1 and H_2 of a group G is again a subgroup of G.

(b) If p is a prime number, the groups $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ are not isomorphic.

(c) There are 25 elements of S_5 of order 2. (Hint: not all of them are transpositions.)