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Spring 1996, Math 185, Section 1

First Midterm

13 February, 1996
12:40-2:00 PM

1. Let G be an open subset of $\mathbf{C}$, let f be a function $G \to \mathbf{C}$, and let $z_0 \in G$.
(a) (5 points) Define what is meant by $f'(z_0)$.

(b) (20 points) Suppose $G = \mathbf{C}$, and f is the function given by $f(z) = \mathrm{Im}(z)$. Show directly from the definition that $f'(z_0)$ does not exist for any z_0. (If you don't see how to do this from the definition, then for partial credit, get the same conclusion by any method.)

2. Given a real-differentiable function $f: \mathbf{C} \to \mathbf{C}$, written as $f = u + iv$, where u and v are real-valued functions, let us define $\partial f/\partial x = \partial u/\partial x + i\partial v/\partial x$ and $\partial f/\partial y = \partial u/\partial y + i\partial v/\partial y$, each a function $\mathbf{C} \to \mathbf{C}$.
(a) (9 points) Write down the Cauchy-Riemann equations for f, and show that these are equivalent to the single equation $\partial f/\partial y = i\partial f/\partial x$.

(b) (9 points) Suppose that the function f discussed above is a polynomial in x and y of degree $\leq d$; i.e., that $f(x+yi) = \Sigma\, a_{m,n} x^m y^n$ for some complex numbers $a_{m,n}$, where m, n range over all nonnegative integer values such that $m+n \leq d$. Using the result of (a), find necessary and sufficient conditions on these coefficients $a_{m,n}$ for f to be holomorphic.

(c) (7 points) Show that in the situation of part (b), if f is holomorphic and the coefficients $a_{m,0}$ are all zero, then $f = 0$.

(d) (10 points) In the situation of (b) above, let $b_m = a_{m,0}$ $(m = 1, \ldots, d)$. Prove that if f is holomorphic, then $f(z) = \Sigma\, b_m z^m$. (Hint: See whether you can apply (c) to the function $g(z) = f(z) - \Sigma\, b_m z^m$.)

3. (25 points) Recall that a *linear fractional transformation* means a map φ from the extended complex plane to the extended complex plane of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c, d are complex numbers such that $ad - bc \neq 0$; and that, to fill in the cases in which that formula does not apply, we define $\varphi(z)$ to be ∞ if $cz+d = 0$, and to be a/c if $z = \infty$.

The following result is proved in the text: *Given three distinct elements* z_1, z_2, z_3 *of the extended complex plane, and likewise three distinct elements* w_1, w_2, w_3, *there exists a unique linear fractional transformation* φ *such that* $\varphi(z_1) = w_1$, $\varphi(z_2) = w_2$, $\varphi(z_3) = w_3$.

Prove this result, with the following modifications: first, for brevity, assume that $z_1, z_2, z_3, w_1, w_2, w_3$ are all complex numbers (i.e., that none is ∞). Second, omit the proof of *uniqueness*. Thirdly, the proof in Sarason justifies one step with the words "because the linear fractional transformations form a group", which I replaced in class by an explicit argument. Your proof should likewise show the argument explicitly.

4. (15 points) Let a and b be nonzero complex numbers. Recall that "a value of a^b" means a complex number of the form $\exp(bq)$, where q is a value of $\log a$.

Show that if w and z are values of a^b, then w^2/z is also a value of a^b.