Math H54 | Honors Linear Algebra and Differential Equations | Spring, 2004

Prof. Haiman

Final Exam

Name ____________________

Instructions:

- There are 10 questions on four pages (both sides).
- Answer in the space provided.
- Point values are indicated on each problem. There are 100 total points.
- No books, notes, calculators or other aids are permitted.
- Please wait until the signal is given to begin before looking at the questions.

1. [10 pts] For the matrix A shown below, find a factorization $PA = LU$, where P is a permutation matrix, L is lower unit triangular, and U is in row echelon form.

$$A = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 2 & -2 & 2 & -1 \\ 1 & -1 & 2 & -3 \end{bmatrix}.$$

2. [8 pts] Suppose A is a 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix},$$

and $\operatorname{tr}(A) = 10$. Find $\det(A)$.

3. [8 pts] Suppose A is an $m \times n$ matrix, B is an $n \times k$ matrix, and $AB = \mathbf{0}$. What can you say about how $\operatorname{rank}(A)$ and $\operatorname{rank}(B)$ are related?

4. Let $P_{\leq 5}$ be the vector space of polynomials of degree less than or equal to 5. Let $V \subseteq P_{\leq 5}$ be the subset consisting of polynomials $p(x)$ such that $p(0) = p(1) = 0$.

(a) [6 pts] Show that V is a subspace of $P_{\leq 5}$.

(b) [6 pts] Find the dimension of V and a basis of V.

5. (a) [6 pts] Compute the angle between the two vectors $\begin{bmatrix} 2 & 1 & 1 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 2 & 1 & -2 & 1 \end{bmatrix}^T$ in $\mathbb{R}^4$, with the usual Euclidean inner product.

(b) [6 pts] Find an orthonormal basis of the subspace spanned by these two vectors.

6. (a) [6 pts] Solve the initial value problem

$$x'(t) + \frac{1}{t+1}x(t) = \frac{2t+1}{t+1}; \quad x(0) = 3.$$

(b) [4 pts] For what values of t is the solution valid?

7. [10 pts] Find all solutions of the differential equation

$$x^{\mathrm{iv}}(t) + 2x''(t) + x(t) = 0.$$

8. Consider the system of linear differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & c \\ 1 & -1 \end{bmatrix} x(t).$$

(a) [5 pts] For which values of the constant c does every solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ have the property that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$?

(b) [5 pts] For which values of c does there exist a non-zero solution such that $x_1(t) = 0$ for infinitely many values of t?

9. [10 pts] Solve the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} \mathbf{x}(t); \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

10. [10 pts] Find one solution of the system of linear differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sin t \\ 0 \end{bmatrix}.$$