MATH 121B: MIDTERM 1, SPRING, 2000

Total score: 100 points.

Problem 1. *In spherical coordinates arclength is given by*

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

(i) *(10 points) Set up the Euler-Lagrange equation for a geodesic on the sphere $r = R$, R a constant.*

(ii) *(10 points) Find the first integral of these equations, and write it in the form $d\phi/d\theta = \ldots$.*

(iii) *(5 points) Show that the constant functions ϕ, i.e. $\phi(\theta) = \phi_0$, ϕ_0 a constant, solve these equations. What curves are these solutions on the sphere?*

Problem 2. *(20 points) Among all functions $y = y(x)$ with $y(x_1) = y_1$, $y(x_2) = y_2$, and $\int_{x_1}^{x_2} y(x)\, dx = A$ (y_1, y_2, A are given constants), find the one that minimizes $\int_{x_1}^{x_2} (y'(x))^2\, dx$. You do not need to find the constants of integration in terms of the given constants.*

Problem 3. *A particle of mass m is moving on a paraboloid $x^2 + y^2 = z$ under the influence of gravitation (z is the vertical direction).*

(i) *(10 points) Find the Lagrangian $L = T - V$ corresponding to the particle. (Recall that in cylindrical coordinates arclength is $ds^2 = dr^2 + r^2\, d\theta^2 + dz^2$.)*

(ii) *(10 points) Write down the corresponding Euler-Lagrange equations.*

(iii) *(5 points) Show that angular momentum is conserved, i.e. $mr(t)^2\dot{\theta}(t)$ is constant.*

Problem 4. *Consider the ODE $-u'' = f$ on the interval $[0, 3]$ with boundary conditions $u(0) = 0 = u(3)$, where f is the function $f(x) = 6x$.*

(i) *(5 points) Write down a weak form of this ODE.*

(ii) *(15 points) Solve the ODE approximately using the method of finite elements. To do so, divide up the interval into 3 subintervals of equal length, and use continuous, piecewise linear trial functions and test functions in the weak form of the ODE.*

You may use, without calculating this, that on an element of length h with endpoints A and B, and associated linear trial functions T_A, T_B (so $T_A(A) = 1$, $T_A(B) = 0$, etc.), the elementary K matrix is

$$k_e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and you may approximate the elementary forcing vector by

$$F_e = \frac{hf(C)}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where C is the midpoint of A and B.

(iii) *(5 points) Graph the approximate solution.*

(iv) *(5 points) Solve the ODE exactly and compare the numerical solution to the exact solution.*