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Fall 2001, Math 113, Sec. 3

Second Midterm

2 Nov., 2001
11:10-12:00

1. (32 points, 8 points apiece) Complete the following definitions. In defining any term, you may use terms defined before it in the text.

(a) If $\varphi: G \to H$ is a homomorphism of groups, then the kernel of φ (denoted $\ker(\varphi)$) means

(b) If G is a group, then the center of G (denoted $Z(G)$) means

(c) If V is a vector space, then a basis of V means

(d) If F is a field and $f(x)$, $g(x)$ are nonzero polynomials over F, then the greatest common divisor of $f(x)$ and $g(x)$ (denoted $\gcd(f(x), g(x))$) means the unique monic polynomial $a(x)$ such that

2. (36 points; 9 points each.) For each of the items listed below, either *give an example*, or give a brief reason why *no example exists*. (If you give an example, you do *not* have to prove that it has the property stated.)

(a) A polynomial $f(x) \in \mathbf{Q}[x]$ which is reducible over $\mathbf{Q}$, but has no root in $\mathbf{Q}$.

(b) A polynomial $f(x) \in \mathbf{Z}[x]$ which has a root in $\mathbf{Q}$ but no root in $\mathbf{Z}$.

(c) Two nonisomorphic abelian groups of order 10.

(d) Groups G and H, a homomorphism $\varphi: G \to H$, and an element $g \in G$ such that g has infinite order, and $\varphi(g)$ has order 3.

3. (12 points) Suppose G is a group which acts on a set S, and s is an element of S. Recall that G_s denotes $\{g \in G \mid gs = s\}$.

Prove that for any $a \in G$ one has $G_{as} = a\,G_s\,a^{-1}$.

4. (20 points) Let G be a group and N a normal subgroup of G. Show that G/N is abelian if and only if for all $a, b \in G$, one has $aba^{-1}b^{-1} \in N$.

(If you correctly derive a necessary and sufficient condition close to this, but do not succeed in transforming it to precisely the above form, you will get appropriate partial credit.)